Discrete Comput Geom OF1–OF21 (2006)
DOI: 10.1007/s00454-006-1251-6

Discrete & Computational
Geometry

Isometry-Invariant Valuations on Hyperbolic Space*

Daniel A. Klain

Department of Mathematical Sciences, University of Massachusetts Lowell,
Lowell, MA 01854, USA
dklain@cs.uml.edu

Abstract. Hyperbolic area is characterized as the *unique* continuous isometry-invariant simple valuation on convex polygons in $\mathbb{H}^2$. We then show that continuous isometry-invariant simple valuations on polytopes in $\mathbb{H}^{2n+1}$ for $n \geq 1$ are determined uniquely by their values at ideal simplices. The proofs exploit a connection between valuation theory in hyperbolic space and an analogous theory on the Euclidean sphere. These results lead to characterizations of continuous isometry-invariant valuations on convex polytopes and convex bodies in the hyperbolic plane $\mathbb{H}^2$, a partial characterization in $\mathbb{H}^3$, and a mechanism for deriving many fundamental theorems of hyperbolic integral geometry, including kinematic formulas, containment theorems, and isoperimetric and Bonnesen-type inequalities.

0. Introduction

A *valuation* on polytopes, convex bodies, or more general class of sets, is a *finitely additive* signed measure; that is, a signed measure that may not behave well (or even be defined) when evaluated on infinite unions, intersections, or differences. A more precise definition is given in the next section. Examples of isometry-invariant valuations on Euclidean space include the Euler characteristic, mean width, surface area, and volume (Lebesgue measure) [KR], [McM3]. Other important valuations on convex bodies and polytopes include projection functions and cross-section measures [Ga], [KR], [Sc1], affine surface area [Lut2], [Lut1], and Dehn invariants [Sah]. Unlike the countably additive measures of classical analysis, which are easily characterized using well-established tools such as the total variation norm, Jordan decomposition, and the Riesz representation theorem [Ru], valuations form a more general class of set functionals that has so far resisted such sweeping classifications [KR], [McM3].

The study of valuations on hyperbolic polytopes is motivated in part by the characterization of many classes of valuations on polytopes and compact convex sets in

* This research was supported in part by NSF Grant #DMS-9803571.

Euclidean space. Such characterizations have had fundamental impact in convex, integral, and combinatorial geometry [Al1]–[Al3], [Ha], [KR], [Kl2], [Kl3], [Lud], [LR], [Sc2], [McM1]–[McM3] as well as to the theory of dissection of polytopes [Bo], [Ha], [KR], [McM3], [Sah].

The fundamental theorem of invariant valuation theory, *Hadwiger's characterization theorem*, classifies *all* continuous isometry-invariant valuations on convex bodies in $\mathbb{R}^n$ as consisting of the linear span of the quermassintegrals (or, equivalently, of McMullen's intrinsic volumes [McM3]):

Theorem 0.1 (Hadwiger). *Suppose that φ is a continuous valuation on compact convex sets in $\mathbb{R}^n$, and that φ is invariant under Euclidean isometries. Then there exist $c_0, c_1, \ldots, c_n \in \mathbb{R}$ such that*

$$\varphi(K) = \sum_{i=0}^{n} c_i V_i(K),$$

for all compact convex sets $K \subseteq \mathbb{R}^n$. In particular, if φ vanishes on sets K having dimension less than n, then φ is proportional to n-dimensional Euclidean volume V_n.

Here the functionals V_i denote extensions of i-dimensional volume to continuous valuations on bodies in $\mathbb{R}^n$, with suitable normalizing constants, so that each V_i is equal to i-volume when restricted to i-dimensional flats in $\mathbb{R}^n$. In particular, V_n denotes volume in $\mathbb{R}^n$, V_{n-1} is one-half of the surface area, and so on down to the (renormalized) mean width V_1 and the Euler characteristic V_0. Hadwiger presented this theorem in [Ha]; alternative shorter proofs can be found in [Kl1] and [KR].

There remain a great many questions regarding how aspects of the Brunn–Minkowski theory of convex bodies (and polytopes) in Euclidean space can be extended to spaces having curvature. Even for spaces of constant curvature, such as the sphere and hyperbolic space, there are many unanswered questions, although work is being done to fill the gap (see, for example, [Fu], [GHS1], [GHS2], and [Ho]).

In particular, little is yet known about *invariant valuations* on polytopes and convex sets in non-Euclidean spaces. A spherical analogue of Theorem 0.1, while plausible for the n-sphere $\mathbb{S}^n$, remains an open question for $n \geq 3$ (see, for example, [KR]). It also remains an open question whether a version of Theorem 0.1 holds for valuations defined only on *polytopes* in Euclidean space (as opposed to the larger class of compact convex sets) [McM3], [MS].

The present article characterizes *hyperbolic* area as the unique continuous isometry-invariant valuation on hyperbolic polygons that vanishes when restricted to points and lines. While in the hyperbolic plane $\mathbb{H}^2$ all ideal triangles (triangles with all three vertices at infinity) are isometrically congruent, this is not true for ideal simplices in higher-dimensional hyperbolic spaces. In odd-dimensional spaces $\mathbb{H}^{2n+1}$ we show that hyperbolic volume is the unique continuous isometry-invariant valuation on hyperbolic polygons that vanishes on lower-dimensional polytopes and agrees with volume on all ideal simplices (i.e., simplices having all vertices at infinity). More precisely, we show that any continuous isometry-invariant *simple* valuation (i.e., vanishing on lower-dimensional sets) is determined uniquely by its values on ideal simplices. It is assumed throughout that the valuations in question are defined on all compact polyhedra in $\mathbb{H}^n$ as well as those having a finite set of vertices at infinity, although valuations are permitted to take infinite values on non-compact sets. These theorems then provide partial analogues to Hadwiger's Theorem 0.1 for the hyperbolic plane $\mathbb{H}^2$.

In analogy to the Euclidean case [Ha], [KR], [San], we also indicate briefly some consequences of valuation characterizations to integral geometry in hyperbolic space. In particular, the *principal kinematic formula* [KR], [San] for the Euler characteristic of a random intersection is generalized to a kinematic formula for arbitrary isometry-invariant valuations on $\mathbb{H}^2$.

The main theorems of this article are indexed as follows:

Valuation characterization theorems:

Theorem 2.1. Area in $\mathbb{H}^2$.
Theorem 3.2. Continuous invariant valuations on $\mathbb{H}^2$.
Theorem 5.3. Finite invariant simple valuations on $\mathbb{H}^{2n+1}$.
Theorem 5.5. Continuous invariant simple valuations on $\mathbb{H}^{2n+1}$.
Corollary 5.6. Continuous invariant valuations on $\mathbb{H}^3$.

Kinematic formulas and other consequences:

Corollary 4.1. Valuation proof of the Gauss–Bonnet formula in $\mathbb{H}^2$.
Theorem 4.3. Kinematic formula for continuous invariant valuations on $\mathbb{H}^2$.
Corollary 4.4. Principal kinematic formula for $\mathbb{H}^2$.
Corollary 4.5. Area formula for parallel bodies in $\mathbb{H}^2$.

1. Convexity in Hyperbolic Space

Let $\mathbb{H}^n$ denote n-dimensional hyperbolic space; that is, the open upper half-space of $\mathbb{R}^n$,

$$\mathbb{H}^n = \{(x_1, \dots, x_n) \mid x_n > 0\}$$

endowed with the hyperbolic distance metric. Recall that hyperplanes (flats) in $\mathbb{H}^n$ correspond to Euclidean hemispheres and half-hyperplanes that are orthogonal, in the Euclidean sense, to $\mathbb{R}^{n-1}$. See, for example, either of [An] or [St].

We denote by $\mathbb{D}^n$ the disk (n-ball) model of hyperbolic space. (See again [An] or [St].)

A set $P \subseteq \mathbb{H}^n$ is a *convex polytope* if P can be expressed as a finite intersection of half-spaces so that P is either compact or has at most a finite number of points (vertices) that lie at infinity (on the boundary of $\mathbb{H}^n$ or $\mathbb{D}^n$). Denote by $\mathcal{P}(\mathbb{H}^n)$ the set of all convex polytopes in $\mathbb{H}^n$. A *polytope* is a finite union of convex polytopes.

More generally, a set $K \subseteq \mathbb{H}^n$ is called *convex* if any two points of K can be connected by a hyperbolic line segment inside K. Let $\mathcal{K}(\mathbb{H}^n)$ denote the union of $\mathcal{P}(\mathbb{H}^n)$ with the set of all *compact* convex sets in $\mathbb{H}^n$. Note that the only non-compact sets in $\mathcal{K}(\mathbb{H}^n)$ are the convex polytopes having a finite collection of vertices at infinity. If $K \in \mathcal{K}(\mathbb{H}^n)$ is non-compact, then K can be expressed in the form

$$K = C * L = \text{ convex hull of } C \cup L,$$

where $C \subseteq \mathbb{H}^n$ is a compact convex set and $L \subseteq \partial\mathbb{H}^n$ is a finite set of points at infinity.

Let $\mathcal{S}$ be a family of subsets of $\mathbb{H}^n$ closed under intersection and containing the empty set $\emptyset$. A function $\varphi : \mathcal{S} \longrightarrow \mathbb{R} \cup \{\pm\infty\}$ is called a *valuation* on $\mathcal{S}$ if, for $K, L \in \mathcal{S}$:

(1) $\varphi(\emptyset) = 0$.
(2) $\varphi(K) \in \mathbb{R}$ (that is, $\varphi(K) \neq \pm\infty$) whenever K is compact.
(3) If $K \cup L \in \mathcal{S}$ as well, and if at least three of the values $\varphi(K)$, $\varphi(L)$, $\varphi(K \cap L)$, and $\varphi(K \cup L)$ are finite, then

$$\varphi(K \cup L) + \varphi(K \cap L) = \varphi(K) + \varphi(L).$$

This third condition is known as the *inclusion–exclusion identity* for valuations.

A valuation φ on $\mathcal{S}$ is called *invariant* if $\varphi(gK) = \varphi(K)$ for all isometries g of $\mathbb{H}^n$ such that $K, gK \in \mathcal{S}$. A valuation φ on $\mathcal{S}$ is called *finite* if $\varphi(K) \in \mathbb{R}$ (that is, $\varphi(K) \neq \pm\infty$) for all $K \in \mathcal{S}$. A valuation φ on a family of sets in $\mathbb{H}^n$ is *simple* if φ vanishes on sets of dimension strictly less than n.

We apply the notion of Hausdorff topology on compact subsets of $\mathbb{H}^n$ with respect to hyperbolic distance.

Definition 1.1. If $C_m \in \mathcal{K}(\mathbb{H}^n)$ is a sequence of compact sets, and if $\lim_{m\to\infty} C_m = C$ in the Hausdorff topology, then we say $C_m \to C$.

More generally, suppose that $C_m \in \mathcal{K}(\mathbb{H}^n)$ is a sequence of compact sets and that L is finite subset of $\partial\mathbb{H}^n$. Let $K_m = C_m * L$, the convex hull. Then we say that $K_m \to K$ iff $C_m \to C$.

Using this definition we can now define *continuous* valuations on $\mathcal{P}(\mathbb{H}^n)$ and $\mathcal{K}(\mathbb{H}^n)$.

Definition 1.2. A real-valued function φ on $\mathcal{P}(\mathbb{H}^n)$ (resp. $\mathcal{K}(\mathbb{H}^n)$) is called *continuous* iff

$$\varphi(K) = \lim_{m\to\infty} \varphi(K_m),$$

whenever $K_i \to K$ in $\mathcal{P}(\mathbb{H}^n)$ (resp. $\mathcal{K}(\mathbb{H}^n)$).

The following is an adaptation of a theorem of Groemer [Gr] for $\mathbb{H}^n$ and for the Euclidean sphere $\mathbb{S}^n$.

Theorem 1.3 (Groemer's Extension Theorem).

(i) *A valuation φ defined on convex polytopes in $\mathbb{H}^n$ (resp. $\mathbb{S}^n$) admits a unique extension to a valuation on the lattice of all polytopes in $\mathbb{H}^n$ (resp. $\mathbb{S}^n$).*
(ii) *A continuous valuation φ on compact convex sets in $\mathbb{H}^n$ (resp. $\mathbb{S}^n$) admits a unique extension to a valuation on the lattice of finite unions of compact convex sets in $\mathbb{H}^n$ (resp. $\mathbb{S}^n$).*

In each case the extension of φ to finite unions is defined by suitable iteration of the inclusion–exclusion identity.

The proof of Theorem 1.3 for the Euclidean case (Groemer's original result) goes through for $\mathbb{H}^n$ and $\mathbb{S}^n$ without essential change, since the original proof is based not on the geometry of $\mathbb{R}^n$, but rather on the algebra of indicator functions and the fact that a polytope is the intersection of half-spaces in $\mathbb{R}^n$, a property that carries over analogously to convex polytopes in $\mathbb{H}^n$ and $\mathbb{S}^n$ as well. For the details of Groemer's original proof, see [Gr] or [KR, p. 44].

In the arguments that follow, the unique extension of valuation φ given by Theorem 1.3 allows us to consider the value of φ on all finite unions of convex polytopes (or compact convex sets in $\mathbb{H}^n$ or $\mathbb{S}^n$), whether or not such unions are actually convex.

Moreover, Definition 1.2 allows us to consider the continuity of valuations at non-compact polytopes having vertices at infinity. Note, however, that only converging sequences of *convex* polytopes (or bodies) are considered. A continuous valuation on $\mathcal{P}(\mathbb{H}^n)$ (resp. $\mathcal{K}(\mathbb{H}^n)$) may not necessarily respect convergent sequences of non-convex sets. (Surface area and perimeter generally do not, for example, even in the Euclidean context.)

2. A Characterization for Hyperbolic Area

We now turn our attention to the two-dimensional space $\mathbb{H}^2$. Important examples of continuous invariant valuations on $\mathcal{P}(\mathbb{H}^2)$ and $\mathcal{K}(\mathbb{H}^2)$ include *hyperbolic area*, denoted A, *hyperbolic perimeter* P, the *Euler characteristic* χ, and a related functional χ_∞, to be defined later.

The perimeter $P(K)$ of a convex region K with non-empty interior in $\mathbb{H}^2$ is given by the length of the boundary ∂K in the hyperbolic metric. If K is a one-dimensional convex set (i.e., a line segment) then $P(K)$ is equal to *twice* the length of K. This normalization guarantees that the perimeter functional is a continuous valuation on $\mathcal{K}(\mathbb{H}^2)$.

The Euler characteristic $\chi(K)$ of a *compact* set $K \in \mathcal{K}(\mathbb{H}^n)$ is defined by $\chi(K) = 1$ if $K \neq \emptyset$, while $\chi(\emptyset) = 0$. Theorem 1.3 ensures that χ has a unique continuous and invariant extension to all finite unions of compact convex sets in $\mathcal{K}(\mathbb{H}^n)$. This extension agrees with the usual definition of Euler characteristic on cell complexes [Mu, p. 124]. In particular, it follows that a compact polygon P decomposed into f_0 vertices, f_1 edges, and f_2 triangles has Euler characteristic $\chi = f_0 - f_1 + f_2$.

We can extend χ to non-compact sets $K \in \mathcal{K}(\mathbb{H}^n)$ as follows. If a convex set $K \in \mathcal{K}(\mathbb{H}^n)$ has exactly m points at infinity, that is, m limit points on $\partial\mathbb{H}^n$, then define

$$\chi(K) = 1 - m.$$

Once again χ extends to a valuation on finite unions of (possibly non-compact) sets in $K \in \mathcal{K}(\mathbb{H}^n)$

This extension of χ to polytopes having vertices at infinity is not unique. We will always denote by χ the extension just given. All other possibilities are accounted for by introducing a related (but distinct) invariant valuation χ_∞ that is special to the hyperbolic case (and does not appear in the Euclidean or spherical contexts). This functional is defined and used in Section 3.

It has been shown that every continuous invariant valuation on polygons in the Euclidean plane (or 2-sphere) is a linear combination of the valuations χ, P, and A (see [Ha] for the Euclidean case or [KR, p. 156] for both the Euclidean and spherical cases). In Section 3 we prove an analogous theorem for the hyperbolic plane: that every continuous invariant valuation on $\mathcal{P}(\mathbb{H}^2)$ (or $\mathcal{K}(\mathbb{H}^2)$) is a linear combination of the valuations χ, χ_∞, P, and A. As a fundamental step towards the results of Section 3, we first prove a characterization theorem for hyperbolic area.

Recall that a valuation φ on $\mathcal{P}(\mathbb{H}^2)$ is *simple* if φ vanishes on points and on all one-dimensional sets.

Theorem 2.1 (Area Characterization Theorem). *Suppose that φ is a continuous invariant simple valuation on $\mathcal{P}(\mathbb{H}^2)$. Then there exists $c \in \mathbb{R}$ such that $\varphi(K) = cA(K)$, for all $K \in \mathcal{P}(\mathbb{H}^2)$.*

It will be seen that continuity is a necessary condition for Theorem 2.1 to hold (see note after Proposition 2.3). Before the proof of Theorem 2.1, we consider some preliminary cases.

Proposition 2.2. *Suppose that φ is a continuous invariant simple valuation on $\mathcal{P}(\mathbb{H}^2)$. Then φ is finite on $\mathcal{P}(\mathbb{H}^2)$.*

Proof. From the definition of valuations we know that $\varphi(K)$ is finite whenever K is compact.

Suppose that T is a triangle in $\mathbb{H}^2$ having exactly one vertex at infinity, such as $\triangle ACX$ in Fig. 1, and suppose that $\varphi(T) = \pm\infty$.

Let M denote the midpoint of A and C. Note that $\triangle ACX = \triangle AMX \cup \triangle MCX$, while $\triangle AMX \cap \triangle MCX = \overline{MX}$, a hyperbolic ray.

Suppose that $\varphi(\triangle AMX)$ and $\varphi(\triangle MCX)$ are both finite. Because φ is simple, we have $\varphi(\overline{MX}) = 0$. The inclusion–exclusion identity would then yield

$$\varphi(T) = \varphi(\triangle ACX) = \varphi(\triangle AMX) + \varphi(\triangle MCX) - \varphi(\overline{MX}),$$

a finite value, contradicting our assumption that $\varphi(T) = \pm\infty$. In other words, if $\varphi(T) = \pm\infty$ then either $\varphi(\triangle AMX) = \pm\infty$ or $\varphi(\triangle MCX) = \pm\infty$ as well. Without loss of generality suppose that $\varphi(\triangle AMX) = \pm\infty$.

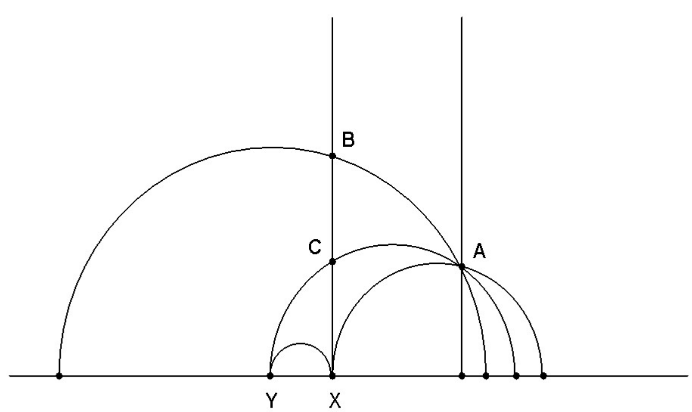

Fig. 1. A hyperbolic triangle is a difference of semi-ideal triangles.

Iterating this procedure yields a sequence of points $M_1, M_2, \ldots \to A$ such that $\varphi(\triangle AM_iX) = \pm\infty$. Since these triangles converge to the hyperbolic ray $\overline{AX} = \triangle AAX$, the continuity of φ implies that $\varphi(\overline{AX}) = \pm\infty$. However, this contradicts the fact that φ is a simple valuation, which requires that $\varphi(\overline{AX}) = 0$. It follows that $\varphi(T) \neq \pm\infty$.

We have shown that if T is a triangle in $\mathbb{H}^2$ having exactly one vertex at infinity, then $\varphi(T)$ is finite. More generally, if K is a polygon with (some or all) vertices at infinity, then K can be expressed as a finite union of triangles each having at most one vertex at infinity (using barycentric subdivision, for example). Since φ is simple, $\varphi(K)$ is equal to the sum of φ over these triangular pieces, so that $\varphi(K)$ is also finite. $\square$

The next proposition is elementary (although useful).

Proposition 2.3. *Suppose that φ is an invariant simple valuation on closed line segments in $\mathbb{R}$, and suppose that $\varphi([\alpha, \beta]) = 0$ for some $\beta > \alpha \in \mathbb{R}$. Then $\varphi(L) = 0$ for all closed line segments $L \subseteq \mathbb{R}$ having length $q(\beta - \alpha)$, where q is any non-negative rational number. If φ is also continuous, then $\varphi(L) = 0$ for all closed line segments $L \subseteq \mathbb{R}$.*

A hyperbolic triangle I is called *ideal* if its three distinct vertices all lie on the line at infinity, or, equivalently, if I has non-empty interior and its three angles each measure zero. Ideal polygons in $\mathbb{H}^2$ are defined similarly. Recall that all ideal hyperbolic triangles in $\mathbb{H}^2$ are isometrically congruent [St, p. 100].

A hyperbolic triangle S is called *semi-ideal* if two of its three vertices lie on the line at infinity. If a semi-ideal triangle S has single non-zero angle θ, then we typically denote it by S_θ. Recall that two semi-ideal hyperbolic triangles in $\mathbb{H}^2$ having the same non-zero angle θ are isometrically congruent.

Proposition 2.4. *Suppose that φ is an invariant simple valuation on $\mathcal{P}(\mathbb{H}^2)$, and suppose that $\varphi(S_\theta) = 0$ for all semi-ideal triangles $S_\theta \subseteq \mathbb{H}^2$. Then $\varphi(T) = 0$ for all triangles $T \in \mathcal{P}(\mathbb{H}^2)$.*

Proof. To begin, suppose that a hyperbolic triangle T has one vertex at infinity. (See, for example, $\triangle ACX$ in Fig. 1.) In this case T can be expressed in terms of two semi-ideal triangles S and S', namely $T \cup S = S'$. For example, $\triangle ACX \cup \triangle CYX = \triangle AYX$ in Fig. 1. Since φ is a simple valuation (so that φ vanishes on edges and vertices), we have $\varphi(T \cap S) = 0$, so that

$$\varphi(T) = \varphi(T) + 0 = \varphi(T) + \varphi(S) = \varphi(T \cup S) + \varphi(T \cap S) = \varphi(S') + 0 = 0.$$

More generally, consider a typical hyperbolic triangle T. In this case T can be expressed in terms of triangles T_1, T_2, each having at least one vertex at infinity; that is, $T \cup T_1 = T_2$. For example, $\triangle ABC \cup \triangle ACX = \triangle ABX$ in Fig. 1. Since $\varphi(T_1) = \varphi(T_2) = 0$, it follows that

$$\varphi(T) = \varphi(T) + 0 = \varphi(T) + \varphi(T_1) = \varphi(T \cup T_1) + \varphi(T \cap T_1) = \varphi(T_2) + 0 = 0. \quad \square$$

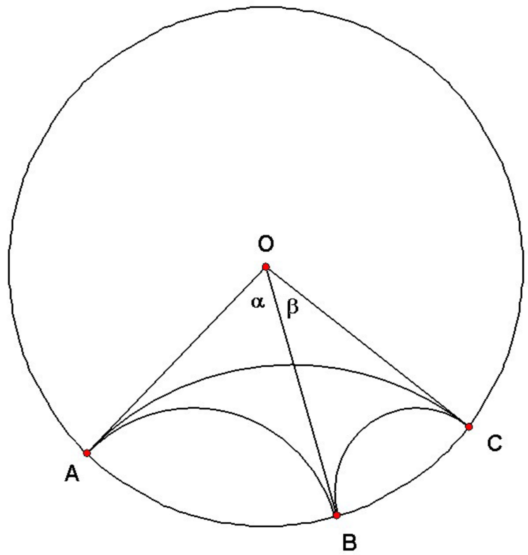

Fig. 2. Arcs in the circle versus semi-ideal hyperbolic triangles.

Proposition 2.5. *Suppose that φ is a continuous invariant simple valuation on $\mathcal{P}(\mathbb{H}^2)$, and suppose that $\varphi(I) = 0$ for all ideal triangles $I \subseteq \mathbb{H}^2$. Then $\varphi(T) = 0$ for all triangles $T \in \mathcal{P}(\mathbb{H}^2)$.*

Proof. Define an invariant simple valuation ψ on closed arcs of the unit circle as follows. Suppose that L_α is a closed arc in $\mathbb{S}^1$ of length α, where $\alpha \in [0, \pi)$. Let S_α be the semi-ideal hyperbolic triangle having non-zero angle α induced by the disc model $\mathbb{D}^2$ for the hyperbolic plane, where we have chosen a fixed base point x_0 as the center of $\mathbb{D}^2$. (See Fig. 2.) Define $\psi(L_\alpha) = \varphi(S_\alpha)$. The function ψ is well-defined since φ is invariant (and since any two semi-ideal triangles having the same non-zero angle α are congruent by some isometry). Moreover, the function ψ is a *valuation* on arcs of the circle, because φ vanishes on ideal triangles. To see this, suppose that two arcs L_α and L_β are adjacent in the circle, having endpoints AB and BC, respectively. Then

$$\begin{aligned}\psi(L_\alpha) + \psi(L_\beta) &= \psi(AB) + \psi(BC) = \varphi(\triangle AOB) + \varphi(\triangle BOC)\\ &= \varphi(\triangle AOC) + \varphi(\triangle ABC) = \psi(AC) + 0 = \psi(L_{\alpha+\beta}),\end{aligned}$$

since $\triangle ABC$ is ideal, as in Fig. 2. Groemer's Theorem 1.3 then implies that ψ has a unique well-defined extension to all finite unions of arcs in $\mathbb{S}^1$, including arcs longer than π.

We now consider the case $\alpha = \pi$. In this instance S_π is a line segment having area zero, so that $\varphi(S_\pi) = 0$ by the simplicity of φ. It now follows from Proposition 2.3 that $\psi(L_{q\pi}) = 0$ for all non-negative rational numbers q.

This argument holds regardless of which base point in $\mathbb{D}^2$ or $\mathbb{H}^2$ we choose to play the role of the center x_0, since a change of center can be accomplished by an isometry. It follows that $\varphi(S_\alpha) = 0$ for any triangle S_α having two vertices at infinity and angle $\alpha = q\pi$ at the remaining vertex in $\mathbb{H}^2$, where q is any rational number. Note that α

varies continuously as the center x_0 is moved along a hyperbolic line in $\mathbb{H}^2$, while the two vertices at infinity remain fixed. (Although this motion is, of course, not an isometry.) Because φ is continuous, it now follows that φ vanishes on *all* semi-ideal triangles. By Proposition 2.4 the valuation φ must vanish on all triangles. □

Proof of Theorem 2.1. Suppose that φ is a continuous invariant simple valuation on $\mathcal{P}(\mathbb{H}^2)$. It follows from Proposition 2.2 that φ is finite. Let I denote an ideal triangle in $\mathbb{H}^2$, and let $c = (1/\pi)\varphi(I)$. Define $\nu(K) = \varphi(K) - cA(K)$ for all $K \in \mathcal{P}(\mathbb{H}^2)$. Recall that an ideal triangle I in $\mathbb{H}^2$ has area π, so that $\nu(I) = 0$. Since the valuation ν satisfies the conditions of Proposition 2.5, it follows that $\nu(T) = 0$ for all triangles $T \in \mathcal{P}(\mathbb{H}^2)$.

For a general hyperbolic polygon $K \in \mathcal{P}(\mathbb{H}^2)$ we can express K as a union of triangles

$$K = T_1 \cup \cdots \cup T_m,$$

where $\dim(T_i \cap T_j) < 2$ for all $i \neq j$. Since ν is simple, it follows that

$$\nu(K) = \sum_{i=1}^{m} \nu(T_i) = 0,$$

for all $K \in \mathcal{P}(\mathbb{H}^2)$, so that

$$\varphi(K) = cA(K),$$

for all $K \in \mathcal{P}(\mathbb{H}^2)$. □

Since any continuous invariant simple valuation φ on $\mathcal{K}(\mathbb{H}^2)$ restricts such a valuation on the *dense* subspace $\mathcal{P}(\mathbb{H}^2)$ of convex polytopes, the continuity of φ, combined with Theorem 2.1, immediately yields the following.

Corollary 2.6. *Suppose that φ is a continuous invariant simple valuation on $\mathcal{K}(\mathbb{H}^2)$. Let I denote an ideal triangle in $\mathbb{H}^2$. Then $\varphi(K) = (\varphi(I)/\pi)A(K)$, for all $K \in \mathcal{K}(\mathbb{H}^2)$.*

3. Invariant Valuations on the Hyperbolic Plane

Theorem 2.1 can be used to characterize *all* isometry-invariant valuations on polygonal regions of the hyperbolic plane. To this end we must first address the one-dimensional case. It is an immediate consequence of Proposition 2.3 that a continuous simple translation-invariant valuation on compact *closed intervals* of $\mathbb{R}$ must be a multiple of Euclidean length. More generally, it follows (or see [KR]) that any continuous translation-invariant valuation on closed intervals of $\mathbb{R}$ must be a linear combination of Euclidean length and the Euler characteristic χ. However, in the hyperbolic plane we must also allow for polygons that have vertices at infinity. In the one-dimensional context this means we must allow for valuations that are defined on *closed rays* as well as closed intervals. Denote by $\mathcal{P}(\mathbb{H}^1)$ the collection of all finite unions and intersections of closed rays and closed intervals in the real line. Note that closed rays have infinite length and Euler characteristic zero (since a closed ray consists of one 0-cell and one 1-cell).

Aside from length and the Euler characteristic χ there is an additional valuation defined on $\mathcal{P}(\mathbb{H}^1)$ that is continuous and translation invariant. Define the valuation χ_∞ on $K \in \mathcal{P}(\mathbb{H}^1)$ by

$$\chi_\infty(K) = \lim_{a\to\infty} \chi(K \cap \{a, -a\}).$$

Since χ is a valuation, it follows that χ_∞ is also a valuation. Evidently $\chi_\infty(K) = 0$ whenever K is a closed interval—indeed, whenever K is compact. Meanwhile $\chi_\infty(K) = 1$ whenever K is a closed ray, while $\chi_\infty(\mathbb{H}^1) = 2$. Evidently χ_∞ is also continuous and isometry invariant.

Proposition 3.1. *Suppose that φ is a continuous isometry invariant valuation on $\mathcal{P}(\mathbb{H}^1)$.*

If φ takes finite values on closed rays, then there exist constants $c_0, c_\infty \in \mathbb{R}$ such that

$$\varphi(K) = c_0\chi(K) + c_\infty\chi_\infty(K),$$

for all $K \in \mathcal{P}(\mathbb{H}^1)$.

If φ takes either value $\pm\infty$ on closed rays, then there exist constants $c_0, c_1 \in \mathbb{R}$ such that

$$\varphi(K) = c_0\chi(K) + c_1 Length(K),$$

for all $K \in \mathcal{P}(\mathbb{H}^1)$.

Proof. Since φ is invariant, φ takes the same value on all singletons. Let $c_0 = \varphi(\{o\})$.

Similarly, since φ is invariant, φ takes the same value on all closed rays. Suppose this is a finite value $c_\infty \in \mathbb{R}$. Write $\mathbb{R}$ as a union of two rays R_1, R_2 (positive and negative) sharing a common endpoint at the origin, $\{o\} = R_1 \cap R_2$. Then

$$\varphi(\mathbb{R}) = \varphi(R_1) + \varphi(R_2) - \varphi(R_1 \cap R_2) = 2c_\infty - c_0.$$

More generally, if C is any closed interval, we can express C is an intersection of two rays R_1, R_2 whose union is all of $\mathbb{R}$, so that

$$\varphi(C) = \varphi(R_1) + \varphi(R_2) - \varphi(\mathbb{R}) = c_\infty + c_\infty - (2c_\infty - c_0) = c_0.$$

It follows that $\varphi(K) = c_0\chi(K) + c_\infty\chi_\infty(K)$ for all $K \in \mathcal{P}(\mathbb{H}^1)$ This completes the proof for the case in which φ takes a finite value on a closed ray.

Suppose instead that φ takes either value $\pm\infty$ on closed rays. Once again, since φ is invariant, φ takes the same value c_0 on all singletons. Since the valuation $\varphi - c_0\chi$ now vanishes on singletons (points), it follows from Proposition 2.3 that $\varphi - c_0\chi$ is a constant multiple of length when applied to closed intervals. In other words, there exists $c_1 \in \mathbb{R}$ such that $\varphi = c_0\chi + c_1 Length$ when applied to finite unions of points and closed intervals. □

The valuation χ_∞ is extended to polygons in $\mathbb{H}^2$ as follows. Choose a base point $x_0 \in \mathbb{H}^2$ and let C_r denote the set of points that lie at a distance $r > 0$ from x_0.

For convex polygons $K \in \mathcal{P}(\mathbb{H}^2)$ define

$$\chi_\infty(K) = \lim_{r\to\infty} \chi(K \cap C_r).$$

Since χ is a valuation, it follows that χ_∞ is also a valuation. Evidently $\chi_\infty(K) = 0$ whenever K is compact, since $K \cap C_r = \emptyset$ for sufficiently large r when K is compact. More generally, for a convex polygon K the value of $\chi_\infty(K)$ is exactly the number of "vertices at infinity" of K. For example, $\chi_\infty = 3$ for ideal triangles, while $\chi_\infty = 1$ for rays and $\chi_\infty = 2$ for hyperbolic lines, generalizing the definition above for the case of $\mathbb{H}^1$. Evidently χ_∞ is independent of the choice of base point x_0. Moreover, for all $K \in \mathcal{K}(\mathbb{H}^2)$ we have $\chi(K) + \chi_\infty(K) = 1$.

Recall that the length functional on the hyperbolic line $\mathbb{H}^1$ extends to the hyperbolic perimeter functional $\frac{1}{2}P$ on polygons in $\mathbb{H}^2$, where the normalization factor of $\frac{1}{2}$ makes the perimeter functional continuous—a line segment in $\mathbb{H}^2$ is a hyperbolic "2-gon", whose perimeter is twice its hyperbolic length, since the line segment can be expressed as the limit of a sequence of flattening triangles.

Theorem 2.1 can now be applied to derive following characterization theorem for continuous invariant valuations on polygons and convex bodies in $\mathbb{H}^2$.

Theorem 3.2 (Invariant Valuation Characterization Theorem for $\mathbb{H}^2$). *Suppose that φ is a continuous invariant valuation on $\mathcal{P}(\mathbb{H}^2)$.*

If φ takes finite values on closed rays, then there exist $c_0, c_2, c_\infty \in \mathbb{R}$ such that

$$\varphi(K) = c_0\chi(K) + c_2A(K) + c_\infty\chi_\infty(K), \tag{1}$$

for all $K \in \mathcal{P}(\mathbb{H}^2)$.

If φ takes either value $\pm\infty$ on closed rays, then there exist $c_0, c_1, c_2 \in \mathbb{R}$ such that

$$\varphi(K) = c_0\chi(K) + c_1P(K) + c_2A(K),$$

for all $K \in \mathcal{P}(\mathbb{H}^2)$.

Since the set $\mathcal{P}(\mathbb{H}^2)$ is dense in $\mathcal{K}(\mathbb{H}^2)$, Theorem 3.2 also holds if $\mathcal{P}(\mathbb{H}^2)$ is replaced with the larger collection $\mathcal{K}(\mathbb{H}^2)$. Theorem 3.2 provides a partial analogue of Hadwiger's Characterization Theorem 0.1, as described in the Introduction (see also [Ha], [Kl1], and [KR]).

Note that Theorem 3.2 may seem incomplete, since it does not appear to account for the valuation $P + \chi_\infty$, for example. However, this is not a problem, since $P + \chi_\infty = P$. Since χ_∞ vanishes on all compact polygonal regions, we can add any scalar multiple of χ_∞ to a valuation having a non-trivial P component without changing the valuation on any $K \in \mathcal{K}(\mathbb{H}^2)$.

Proof of Theorem 3.2. Let φ denote a continuous invariant valuation on $\mathcal{P}(\mathbb{H}^2)$.

Suppose that φ takes finite values on closed rays. By Proposition 3.1 the restriction of φ to a hyperbolic line ℓ has the form $\varphi = c_0\chi + c_\infty\chi_\infty$, where $c_0, c_\infty \in \mathbb{R}$ are constants independent of the choice of hyperbolic line ℓ (because φ is isometry invariant).

It follows that the valuation ν on $\mathcal{P}(\mathbb{H}^2)$ given by

$$\nu = \varphi - c_0\chi - c_\infty\chi_\infty$$

vanishes on all $K \in \mathcal{P}(\mathbb{H}^2)$ of dimension less than 2; that is, ν is a continuous invariant simple valuation on $\mathcal{P}(\mathbb{H}^2)$. Theorem 2.1 then implies the existence of $c_2 \in \mathbb{R}$ such that $\nu(K) = c_2A(K)$ for all $K \in \mathcal{P}(\mathbb{H}^2)$.

Suppose instead that φ takes infinite values on closed rays. A similar argument using Proposition 3.1 yields the analogous result, in which χ_∞ is replaced by the perimeter P. □

Note that valuations of type (1) in Theorem 3.2 are constant on line segments. This provides a simple test for when χ_∞ can be omitted from consideration.

Corollary 3.3. *Suppose that φ is a continuous invariant valuation on $\mathcal{P}(\mathbb{H}^2)$ that is not constant on line segments. Then there exist $c_0, c_1, c_2 \in \mathbb{R}$ such that, for all $K \in \mathcal{P}(\mathbb{H}^2)$,*

$$\varphi(K) = c_0\chi(K) + c_1 P(K) + c_2 A(K).$$

Meanwhile, a finiteness condition will determine when the perimeter P is omitted.

Corollary 3.4. *Suppose that φ is a continuous invariant* finite *valuation on $\mathcal{P}(\mathbb{H}^2)$. Then there exist $c_0, c_2, c_\infty \in \mathbb{R}$ such that, for all $K \in \mathcal{P}(\mathbb{H}^2)$,*

$$\varphi(K) = c_0\chi(K) + c_2 A(K) + c_\infty \chi_\infty.$$

Theorem 3.2 also implies that any continuous invariant valuation on $\mathcal{P}(\mathbb{H}^2)$ is determined up to a multiple of χ_∞ by its values on a hyperbolic disc D_r of radius r. Recall that

$$P(D_r) = 2\pi \sinh r \quad \text{and} \quad A(D_r) = 2\pi(\cosh r - 1). \tag{2}$$

See, for example, p. 85 of [St]. Since $\chi(D_r) = 1$ for all $r \geq 0$, we have

$$\varphi(D_r) = c_0 + 2\pi c_1 \sinh r + 2\pi c_2(\cosh r - 1).$$

The coefficients c_i in Theorem 3.2 are easily computed once $\varphi(D_r)$ is known for three suitable values of r.

4. Integral Geometry in the Hyperbolic Plane

Hadwiger's Characterization Theorem 0.1 for invariant valuations on Euclidean space provides a powerful mechanism for deriving fundamental integral-geometric identities. For a number of applications and consequences of Hadwiger's theorem, see, for example, [Ha] and [KR]. The Area Theorem 2.1 and the equivalent characterization Theorem 3.2 provide similar advantages in the context of hyperbolic integral geometry. A simple though fundamental example is the area formula for hyperbolic triangles and polygons, a special case of the Gauss–Bonnet theorem [St, p. 100].

Corollary 4.1 (Gauss–Bonnet Theorem for Polygons). *Suppose that K is a simple closed polygonal curve in $\mathbb{H}^2$, and suppose that the boundary of K has n vertices (possibly at infinity), with corresponding interior angle measures $\alpha_1, \ldots, \alpha_n \in [0, \pi]$. Then the area of K is given by*

$$A(K) = (n-2)\pi(\chi(K) + \chi_\infty(K)) - \sum_{i=1}^{n} \alpha_i.$$

Note that the χ_∞ term vanishes when K is compact.

Proof. For a *convex* polygon K define $\Sigma(K)$ to be the sum of the angles between unit *outer* normals of K wherever two adjacent edges meet at a vertex. If K and L are convex polygons such that $K \cup L$ is also convex, then the boundaries ∂K and ∂L must meet either at vertices or at edges having the same unit normal, so that $\Sigma(K \cup L) + \Sigma(K \cap L) = \Sigma(K) + \Sigma(L)$. It follows from Groemer's Theorem 1.3 that Σ extends to a valuation on $\mathcal{P}(\mathbb{H}^2)$. Evidently Σ is invariant and continuous. Moreover, $\Sigma = 2\pi$ for all points, line segments, and rays. By Theorem 3.2 there exist $a, b, c \in \mathbb{R}$ such that $\Sigma = a\chi + bA + c\chi_\infty$. Since $\Sigma = 2\pi$ on points, $a = 2\pi$. Since $\Sigma = 2\pi$ on rays, $c = 2\pi$. Because $\Sigma = 3\pi$ on ideal triangles (while $\chi + \chi_\infty = 1$ and $A = \pi$) we obtain $\Sigma = 2\pi(\chi + \chi_\infty) + A$.

Let $\sigma(K)$ denote the sum of the *interior* angles at the vertices of K. (Note that σ is *not* a valuation.) If K has n vertices then $\sigma(K) + \Sigma(K) = \pi n$, so that

$$\begin{aligned} A(K) &= \Sigma(K) - 2\pi(\chi(K) + \chi_\infty(K)) = (\pi n - \sigma(K)) - 2\pi(\chi(K) + \chi_\infty(K)) = \cdots \\ &= (n-2)\pi(\chi(K) + \chi_\infty(K)) - \sum_{i=1}^{n} \alpha_i. \end{aligned}$$ □

Recall that if T is a hyperbolic triangle then $n = 3$ and $\chi(T) + \chi_\infty(T) = 1$.

Corollary 4.2 (Area of a Hyperbolic Triangle). *If a triangle T in $\mathbb{H}^2$ has interior angle measures $\alpha, \beta, \gamma \in [0, \pi]$ then the area of T is then given by the* angle deficit:

$$A(T) = \pi - (\alpha + \beta + \gamma).$$

Theorem 3.2 also yields a quick proof of the *principal kinematic formula* for compact convex sets in $\mathbb{H}^2$, a fundamental theorem of geometric probability [KR], [San]. While a classical proof of the principal kinematic formula can be found in [San, p. 321], Theorem 3.2 immediately implies the following stronger result.

Theorem 4.3 (Kinematic Formula for Invariant Valuations on $\mathbb{H}^2$). *Suppose that φ is a continuous isometry invariant valuation on $\mathcal{P}(\mathbb{H}^2)$ (or $\mathcal{K}(\mathbb{H}^2)$). Then there exists a constant real 4×4 symmetric matrix C such that*

$$\int_g \varphi(K \cap gL)\, dg = \begin{bmatrix} \chi(K) & P(K) & A(K) & \chi_\infty(K) \end{bmatrix} C \begin{bmatrix} \chi(L) \\ P(L) \\ A(L) \\ \chi_\infty(K) \end{bmatrix} \tag{3}$$

for all $K, L \in \mathcal{P}(\mathbb{H}^2)$ (or $\mathcal{K}(\mathbb{H}^2)$).

The integral on the left-hand side of (3) is taken with respect to the hyperbolic area on $\mathbb{H}^2$ and the invariant Haar probability measure on the group G_0 of hyperbolic isometries that fix a base point $x_0 \in \mathbb{H}^2$. To define this more precisely, denote by t_x the unique hyperbolic translation of $\mathbb{H}^2$ that maps x_0 to a point $x \in \mathbb{H}^2$. Then define

$$\int_g \varphi(K \cap gL)\, dg = \int_{x \in \mathbb{H}^2} \int_{\gamma \in G_0} \varphi(K \cap t_x(\gamma L))\, d\gamma\, dx, \tag{4}$$

where we use the probabilistic normalization $\int_{g \in G_0} dg = 1$.

Proof of Theorem 4.3. To begin, define

$$\varphi(K, L) = \int_g \varphi(K \cap gL)\, dg.$$

For fixed K, the set function $\varphi(K, L)$ is a valuation in the variable L; in fact, it is an invariant valuation, since

$$\varphi(K, g_0L) = \int \varphi(K \cap gg_0L)\, dg = \int \varphi(K \cap gL)\, dg,$$

for each isometry g_0.

It follows from Theorem 3.2 that we can express $\varphi(K, L)$ as a linear combination of the valuations χ, P, A, χ_∞, with coefficients $c_i(K)$ depending on K:

$$\varphi(K, g_0L) = c_0(K)\chi(L) + c_1(K)P(L) + c_2(K)A(L) + c_\infty(K)\chi_\infty(L).$$

Meanwhile, for fixed L, the set function $\varphi(K, L)$ is a valuation in the variable K. It follows that each of the coefficients $c_i(K)$ is a valuation in the variable K. Moreover, since the valuation φ and the Haar integral in the left-hand side of (3) are both isometry invariant, we have

$$\varphi(K, L) = \int \varphi(K \cap gL)\, dg = \int \varphi(g^{-1}K \cap L)\, dg = \int \varphi(gK \cap L)\, dg = \varphi(L, K).$$

Therefore, each $c_i(K)$ is an *invariant* valuation in the variable K, so that Theorem 3.2 applies, yielding the matrix equation

$$\varphi(K, L) = \begin{bmatrix} \chi(K) & P(K) & A(K) & \chi_\infty(K) \end{bmatrix} C \begin{bmatrix} \chi(L) \\ P(L) \\ A(L) \\ \chi_\infty(K) \end{bmatrix},$$

where $C = [c_{ij}]_{i,j\in\{0,1,2,\infty\}}$ is a 4×4 matrix of real constants, independent of K and L. Since $\varphi(K, L) = \varphi(L, K)$, it follows that $c_{ij} = c_{ji}$. $\square$

The following special case is of fundamental importance in integral geometry and geometric probability. See, for example, [Fu], [Ho], [KR], [San], and [SW].

Corollary 4.4 (Principal Kinematic Formula for $\mathbb{H}^2$). *For* $K, L \in \mathcal{K}(\mathbb{H}^2)$,

$$\int_g \chi(K \cap gL)\, dg = \chi(K)A(L) + \frac{1}{2\pi}P(K)P(L) + A(K)\chi(L) + \frac{1}{2\pi}A(K)A(L). \quad (5)$$

In order to verify (5) we require the notion of the *parallel body*. The parallel body of K having radius $\varepsilon \geq 0$ is the set K_ε of points in $\mathbb{H}^2$ (or $\mathbb{H}^n$) whose (hyperbolic) Hausdorff distance to the set K is at most ε. See, for example, [Sc1].

Let D_ε denote the set of points that lie at most a distance ε from x_0, where x_0 is a chosen base point. Note that $gD_\varepsilon = D_\varepsilon$ for all isometries $g \in G_0$. When K is a compact convex set, the indicator function of K_ε is given by

$$I_{K_\varepsilon}(x) = \chi(K \cap t_x(D_\varepsilon)) = \begin{cases} 1 & \text{if} \quad x \in K_\varepsilon, \\ 0 & \text{if} \quad x \notin K_\varepsilon, \end{cases}$$

since $t_x(D_\varepsilon)$ is the set of points a distance at most ε from the point x.

Since we have $gD_\varepsilon = D_\varepsilon$ for all isometries $g \in G_0$, the area of the parallel body K_ε is then given by

$$\begin{aligned} A(K_\varepsilon) = \int_{x\in\mathbb{H}^2} I_{K_\varepsilon}\, dx &= \int_{x\in\mathbb{H}^2} \chi(K \cap t_x(D_\varepsilon))\, dx \\ &= \int_{x\in\mathbb{H}^2} \int_{g\in G_0} \chi(K \cap t_x(gD_\varepsilon))\, dg\, dx = \chi(K, D_\varepsilon). \end{aligned} \tag{6}$$

Since χ is a valuation and integration is linear, it follows that the mapping $K \to A(K_\varepsilon)$ is a *valuation* in the parameter K (where ε is a fixed constant).

Proof of Corollary 4.4. In order to compute the values of the coefficients c_{ij}, we evaluate $\chi(K, L)$ by calculating for the cases in which $K = L = D_r$, for some $r \geq 0$. For example, it is evident from (4) that $\chi(K, L) = 0$ when K and L are points, or when K is a point and L is a line segment, so that $c_{00} = c_{10} = c_{01} = 0$. More generally, if L is a point, then $\chi(K, L) = \chi(K, D_0) = A(K) = A(K)\chi(L)$ by (6), so that $c_{0j} = c_{j0} = 0$ for all $j \neq 2$, while $c_{02} = c_{20} = 1$.

Next, note that if K_a is a line segment of length a and $\dim L \geq 1$, then the family of motions of L that meet K_a is strictly increasing as a increases. By Corollary 3.3 the valuation χ_∞ does not appear anywhere in the expression for $\chi(K, L)$, so that $c_{\infty j} = c_{j\infty} = 0$ for all j.

To compute the remaining c_{ij} we use the identity

$$P(D_a)^2 = \pi A(D_{2a}) \qquad \text{for} \quad a \geq 0, \tag{7}$$

an elementary consequence of (2).

Denote by ∂D the boundary of a hyperbolic disk D. Note that $\chi(\partial D) = A(\partial D) = 0$, while $P(\partial D) = 2P(D)$, since the "perimeter" of a one-dimensional curve is twice its length. (Recall that a one-dimensional curve is, in the limiting sense, a "two-sided" polygon.) We now compute

$$\chi(\partial D_a, \partial D_a), \quad \chi(\partial D_a, D_a), \quad \text{and} \quad \chi(D_a, D_a).$$

Since D_a is the set of points which lie at most a distance a from x_0, we have $gD_a = D_a$ for all isometries $g \in G_0$, and similarly for ∂D_a. By (6),

$$\chi(K, D_a) = A(K_a),$$

where K_a is the a-parallel body of K. In particular, $\chi(D_a, D_a) = A(D_{2a})$.

If two closed disks intersect, they do so as a compact set, while their boundaries *generically* intersect in exactly *two points*. In particular, since $\chi(D_a \cap t_x(\partial D_a)) = \chi(D_a \cap t_x(D_a))$ for all $x \neq x_0$, while $\chi(\partial D_a \cap t_x(\partial D_a)) = 2\chi(D_a \cap t_x(D_a))$ for all $x \notin \partial D_{2a}$ with $x \neq x_0$. Therefore

$$\tfrac{1}{2}\chi(\partial D_a, \partial D_a) = \chi(D_a, D_a) = A(D_{2a}) = \frac{1}{\pi}P(D_a)^2, \tag{8}$$

by (7).

Since $\chi(\partial D_a) = A(\partial D_a) = 0$, it follows from (3) and (8) that

$$\frac{2}{\pi}P(D_a)^2 = \chi(\partial D_a, \partial D_a) = c_{11}P(\partial D_a)P(\partial D_a) = 4c_{11}P(D_a)^2,$$

so that $c_{11} = 1/2\pi$. Similar elementary considerations lead to $c_{22} = 1/2\pi$ and $c_{12} = c_{21} = 0$, completing the proof of the kinematic formula (5). □

The following corollary can be derived directly (see p. 322 in [San]), but follows immediately from (6) and Corollary 4.4.

Corollary 4.5 (The Area of a Parallel Body). *For $K \in \mathcal{K}(\mathbb{H}^2)$ and $\varepsilon \geq 0$,*

$$\begin{aligned} A(K_\varepsilon) &= A(D_\varepsilon) + \frac{1}{2\pi}P(D_\varepsilon)P(K) + \left(1 + \frac{1}{2\pi}A(D_\varepsilon)\right)A(K) \\ &= 2\pi(\cosh\varepsilon - 1) + (\sinh\varepsilon)P(K) + (\cosh\varepsilon)A(K). \end{aligned}$$

Corollary 4.5 gives the hyperbolic analogue of *Steiner's formula* for the area (or volume) of a Euclidean parallel body [Sc1]. It would be interesting to see how a suitable variation of Corollary 4.4 (possibly using integration over a suitable chosen proper subgroup of isometries) might yield hyperbolic analogues of Minkowski's *mixed volumes* and the related Brunn–Minkowski theory [Sc1].

Corollary 4.4 and its higher-dimensional generalizations have numerous applications to questions in geometric probability, leading, for example, to hyperbolic analogues of Hadwiger's containment theorem for planar regions [KR], [San] and to Bonnesen's inequality for area (see [Kl3] and [San, p. 120]), a generalization of the classical isoperimetric inequality (see also [Os]). Variations of these kinematic techniques can also be found in [Kl3] and [San, p. 324].

5. Characterizing Valuations in $\mathbb{H}^n$

The proof of the hyperbolic area characterization, Theorem 2.1, relied in part on a relationship between an invariant valuation on $\mathbb{H}^2$ and a derived invariant valuation on the unit circle, which was in turn easily characterized. Equally important was the fact that all ideal triangles in $\mathbb{H}^2$ are congruent with respect to an hyperbolic isometry.

For dimensions $n \geq 3$, ideal simplices in $\mathbb{H}^n$ are no longer necessarily congruent. Moreover, while *spherical area* in the two-dimensional sphere $\mathbb{S}^2$ has a valuation characterization (see p. 156 in [KR]), the analogous characterization of spherical volume in $\mathbb{S}^n$

remains an open conjecture for $n \geq 3$. As a result, the methods of the previous sections do not entirely generalize to higher-dimensional hyperbolic space.

In order to extend some of the previous results to higher-dimensional hyperbolic space, we make do with the following partial result regarding invariant valuations on spherical polytopes in an even-dimensional sphere $\mathbb{S}^{2n}$.

Define a *lune* in $\mathbb{S}^{2n}$ to be a subset of $\mathbb{S}^{2n}$ consisting of the intersection of at most $2n$ hemispheres.

Theorem 5.1. *Suppose that φ is an isometry-invariant simple valuation on $\mathcal{P}(\mathbb{S}^{2n})$. If $\varphi(L) = 0$ for all lunes $L \subseteq \mathbb{S}^{2n}$, then $\varphi(K) = 0$ for all $K \in \mathcal{P}(\mathbb{S}^{2n})$.*

Theorem 5.1 will play a role analogous to that of Proposition 2.3 characterizing hyperbolic volume in higher dimensions. Note that continuity plays no role in this theorem. Theorem 5.1 can be found on p. 165 in [KR]. For completeness we include a proof here.

Proof of Theorem 5.1. Suppose that Δ is a spherical simplex in $\mathbb{S}^{2n}$ given by the intersection of hemispheres $\Delta = H_1 \cap \cdots \cap H_{2n+1}$.

For $X \subseteq \mathbb{S}^{2n}$, denote by X^c the closure of the complement $\mathbb{S}^{2n} - X$. Note that

$$\left(\bigcup_{i=1}^{2n+1} H_i \right)^c = \bigcap_{i=1}^{2n+1} H_i^c = -\Delta.$$

Because φ vanishes on lunes, $\varphi(\mathbb{S}^{2n}) = 0$. Since φ is simple and invariant,

$$\varphi\left(\bigcup_{i=1}^{2n+1} H_i \right) = \varphi(\mathbb{S}^{2n}) - \varphi\left(\left(\bigcup_{i=1}^{2n+1} H_i \right)^c \right) = 0 - \varphi(-\Delta) = -\varphi(\Delta). \tag{9}$$

Meanwhile, since φ is a valuation, the inclusion–exclusion identity yields

$$\varphi\left(\bigcup_{i=1}^{2n+1} H_i \right) = \sum_{i=1}^{2n+1} \varphi(H_i) - \sum_{i_1 < i_2} \varphi(H_{i_1} \cap H_{i_2}) + \cdots + (-1)^{2n} \varphi(H_1 \cap \cdots \cap H_{2n+1}). \tag{10}$$

Since φ vanishes on hemispheres and lunes, all of the terms on the right-hand side of (10) vanish except possibly for the last term: $(-1)^{2n}\varphi(H_1 \cap \cdots \cap H_{2n+1}) = \varphi(\Delta)$. Combining (9) and (10) then yields

$$\varphi(\Delta) = \varphi\left(\bigcup_{i=1}^{2n+1} H_i \right) = -\varphi(\Delta), \tag{11}$$

so that $\varphi(\Delta) = 0$. Since every polytope $K \in \mathcal{P}(\mathbb{S}^{2n})$ can be expressed as a union of spherical simplices intersecting in dimension less than $2n$, it follows that $\varphi(K) = 0$ for all $K \in \mathcal{P}(\mathbb{S}^{2n})$. $\square$

The sign discrepancy in (11), which in turn implies that $\varphi = 0$ identically, depends on the fact that the inclusion–exclusion expansion on the right-hand side of (10) terminates with an even power of -1, a consequence of the *even*-dimensionality of $\mathbb{S}^{2n}$. A version of Theorem 5.1 for $\mathbb{S}^{2n+1}$ remains an open problem.

An hyperbolic n-simplex S is called *semi-ideal* if at least n of its $n+1$ vertices lie on the plane at infinity.

Proposition 5.2. *Suppose that φ is an invariant simple valuation on $\mathcal{P}(\mathbb{H}^n)$, and suppose that $\varphi(S) = 0$ for all ideal and semi-ideal simplices $S \subseteq \mathbb{H}^n$. Then $\varphi(K) = 0$ for all polytopes $K \in \mathcal{P}(\mathbb{H}^n)$.*

Proof. Suppose T is a simplex in $\mathbb{H}^n$ having at least two vertices $x_0 \neq x_1$ that *do not* lie at infinity. Let x^* denote the point at infinity that is collinear with x_0 and x_1, so that x_1 lies between x_0 and x^*. If $T = [x_0, x_1, x_2, \ldots, x_n]$, let T_1 and T_2 denote the simplices defined by $T_1 = [x^*, x_1, x_2, \ldots, x_n]$ and $T_2 = [x^*, x_0, x_2, \ldots, x_n]$. Then $T \cup T_1 = T_2$, while $\dim(T \cap T_1) \leq n-1$. Since φ is a simple valuation, we have

$$\varphi(T) + \varphi(T_1) = \varphi(T \cup T_1) + \varphi(T \cap T_1) = \varphi(T_2) + 0 = \varphi(T_2). \qquad (12)$$

We now proceed by induction on the number of vertices at infinity. If T has $n-1$ or more vertices at infinity, then T is semi-ideal, so that $\varphi(T) = 0$, by our initial assumption. Suppose that $\varphi(T') = 0$ whenever T' has k vertices at infinity, for some $k \geq 1$. If a simplex T has $k-1$ vertices at infinity, then (12) implies that $\varphi(T) = \varphi(T_2) - \varphi(T_1)$, for some T_1, T_2 having k vertices at infinity. Hence, $\varphi(T) = 0$ as well. It follows that $\varphi(T) = 0$ for all n-simplices T.

Since every polytope $K \in \mathcal{P}(\mathbb{H}^n)$ has a finite simplicial decomposition, φ vanishes on $\mathcal{P}(\mathbb{H}^n)$. ☐

For a compact convex set K in $\mathbb{H}^n$, denote by $V_n(K)$ the hyperbolic volume of K.

Theorem 5.3 (Ideal Determination Theorem—Finite Case). *Suppose that ψ_1 and ψ_2 are invariant finite simple valuations on $\mathcal{P}(\mathbb{H}^{2n+1})$, where n is a positive integer. If $\psi_1(I) = \psi_2(I)$ for all ideal $(2n+1)$-dimensional simplices I in $\mathbb{H}^{2n+1}$, then $\psi_1(K) = \psi_2(K)$, for all $K \in \mathcal{P}(\mathbb{H}^{2n+1})$.*

Note: We do not assume that the valuations ψ_i in Theorem 5.3 are continuous.

Proof. Let $\varphi = \psi_1 - \psi_2$, so that $\varphi(I) = 0$ for all ideal I. It now suffices to show that φ is identically zero on all polytopes.

Define an invariant simple valuation η on convex spherical polytopes in $\mathbb{S}^{2n}$ as follows. Choose a base point $x_0 \in \mathbb{H}^{2n+1}$. Suppose that L is a convex spherical polytope in $\mathbb{S}^{2n}$ having extreme points $z_1, \ldots, z_m$ that all lie in the same open hemisphere of $\mathbb{S}^{2n}$. Let Q_L denote hyperbolic convex hull of the points $x_0, z_1, \ldots, z_m$ in $\mathbb{H}^{2n+1}$, where $\mathbb{S}^{2n}$ is viewed as the boundary of the Poincaré ball model $\mathbb{D}^{2n+1}$ for the hyperbolic space $\mathbb{H}^{2n+1}$ having center at x_0. Define $\eta(L) = \varphi(Q_L)$. The function η is an orthogonal invariant since φ is invariant (while isometries of $\mathbb{S}^{2n}$ are restrictions of certain hyperbolic isometries of $\mathbb{H}^{2n+1}$). Moreover, the function η satisfies the inclusion–exclusion condition for valuations, because φ vanishes on ideal simplices. (See, for example, Fig. 2.) Groemer's

Theorem 1.3 then implies that η has a unique well-defined extension to all finite unions of convex spherical polytopes, including those no longer contained in an open hemisphere. The simplicity of η also follows immediately from the simplicity of φ.

If L is a lune in $\mathbb{S}^{2n}$ consisting of the intersection of exactly $2n$ generically positioned hemispheres, then we call L a *minimal lune*. In this case L contains exactly one pair of antipodal points $a, a' \in \mathbb{S}^{2n}$, and $L = L_1 \cup L_2$, where L_1 and L_2 are spherical simplices congruent by a reflection of $\mathbb{S}^{2n}$ across the great-$(2n-1)$-subsphere Z normal to the axis $\overline{aa'}$. Note that Q_{L_1} (resp. Q_{L_2}) is a semi-ideal simplex having one vertex at a (resp. a'), one vertex at x_0, and the remaining vertices in the great subsphere Z. Since a and a' are antipodal points, the union $Q_{L_1} \cup Q_{L_2}$ is an *ideal* simplex having a vertex at each of a and a' and the remaining vertices in Z.

Since $L_1 \cap L_2$ has dimension less than $2n$ in $\mathbb{S}^{2n}$, we have $\eta(L_1 \cap L_2) = 0$. Similarly, since $Q_{L_1} \cap Q_{L_2}$ has dimension less than $2n+1$, we have $\varphi(Q_{L_1} \cap Q_{L_2}) = 0$. It follows that

$$\eta(L) = \eta(L_1) + \eta(L_2) = \varphi(Q_{L_1}) + \varphi(Q_{L_2}) = \varphi(Q_{L_1} \cup Q_{L_2}) + 0 = 0,$$

since φ vanishes on ideal simplices. If L is a lune consisting of an intersection of fewer than $2n$ hemispheres, then L can be subdivided into a finite union of minimal lunes, so that $\eta(L) = 0$ once again, by the simplicity property of η.

Theorem 5.1 then implies that $\eta(K) = 0$ for all $K \in \mathcal{P}(\mathbb{S}^{2n})$. In particular, φ vanishes on all hyperbolic simplices having one vertex at x_0 (or, since our choice of x_0 was arbitrary, at any other point of $\mathbb{H}^{2n+1}$) and all remaining vertices at infinity (i.e., on $\mathbb{S}^{2n}$, using the Poincaré ball model for $\mathbb{H}^{2n+1}$). In other words, φ vanishes on all semi-ideal simplices in $\mathbb{H}^{2n+1}$. It follows from Proposition 5.2 that φ must vanish on *all* of $\mathcal{P}(\mathbb{H}^{2n+1})$. □

The proof of the following proposition is exactly analogous to the proof of Proposition 2.2.

Proposition 5.4. *Suppose that φ is a continuous invariant simple valuation on $\mathcal{P}(\mathbb{H}^n)$. Then φ is finite on $\mathcal{P}(\mathbb{H}^n)$.*

We are now ready to state and prove a partial generalization of the Area Theorem 2.1.

Theorem 5.5 (Ideal Determination Theorem—Continuous Case). *Suppose that ψ_1 and ψ_2 are continuous invariant simple valuations on $\mathcal{P}(\mathbb{H}^{2n+1})$, where n is a positive integer. If $\psi_1(I) = \psi_2(I)$ for all ideal $(2n+1)$-dimensional simplices I in $\mathbb{H}^{2n+1}$, then $\psi_1(K) = \psi_2(K)$, for all $K \in \mathcal{P}(\mathbb{H}^{2n+1})$.*

Proof. Since each of the valuations ψ_i is continuous, invariant, and simple on $\mathcal{P}(\mathbb{H}^{2n+1})$, each ψ_i is also a finite valuation by Proposition 5.4, so that Theorem 5.3 applies, and $\psi_1(K) = \psi_2(K)$ for all $K \in \mathcal{P}(\mathbb{H}^{2n+1})$. □

Note: If each of the valuations ψ_i in Theorem 5.5 is defined on all of $\mathcal{K}(\mathbb{H}^{2n+1})$, then the continuity of the ψ_i and the density of $\mathcal{P}(\mathbb{H}^{2n+1})$ in $\mathcal{K}(\mathbb{H}^{2n+1})$ together imply that

$\psi_1(K) = \psi_2(K)$ for all $K \in \mathcal{K}(\mathbb{H}^{2n+1})$ as well. In other words, continuous invariant simple valuations on $\mathbb{H}^{2n+1}$ are completely determined by their values on ideal simplices.

In the case of dimension 3, we can also remove the condition of simplicity.

Corollary 5.6 (Ideal Determination Theorem for $\mathbb{H}^3$). *Suppose that ψ_1 and ψ_2 are invariant continuous valuations on $\mathcal{P}(\mathbb{H}^3)$. If ψ_1 and ψ_2 agree on singletons* (*points*), *and if* $(\psi_1 - \psi_2)(I) = 0$ *for all ideal simplices I in $\mathbb{H}^3$, then* $(\psi_1 - \psi_2)(K) = 0$, *for all* $K \in \mathcal{K}(\mathbb{H}^3)$.

Evidently the conclusion of Corollary 5.6 implies that $\psi_1 = \psi_2$ as valuations on $\mathcal{K}(\mathbb{H}^3)$. In the hypothesis we require $(\psi_1 - \psi_2)(I) = 0$ rather than $\psi_1(I) = \psi_2(I)$ in order to account more carefully for the possibility that ψ_1 and ψ_2 take infinite values on some ideal simplex.

Proof. Let ξ denote a two-dimensional hyperbolic plane inside $\mathbb{H}^3$. Since the difference $\psi_1 - \psi_2$ vanishes on hyperbolic lines (ideal 1-simplices), it follows from Theorem 3.2 that

$$(\psi_1 - \psi_2)|_\xi = c_0\chi + c_\infty\chi_\infty + c_2 A,$$

where A denotes the hyperbolic area in ξ. Since $\psi_1 - \psi_2$ vanishes on points, it follows that $c_0 = 0$. Since $\psi_1 - \psi_2$ vanishes on hyperbolic lines, $c_\infty = 0$ as well. If I is an ideal triangle inside ξ then

$$0 = (\psi_1 - \psi_2)(I) = c_2 A(I) = c_2\pi,$$

so that $c_2 = 0$. In other words, the invariant valuation $\psi_1 - \psi_2$ vanishes on all polygons in ξ, and therefore in any two-dimensional flat of $\mathbb{H}^3$, so that $\psi_1 - \psi_2$ is *simple*.

Theorem 5.5 now applies to the simple valuation $\psi_1 - \psi_2$. Since $(\psi_1 - \psi_2)(I) = 0$ for all three-dimensional ideal simplices in $\mathbb{H}^3$, it follows that $(\psi_1 - \psi_2)(K) = 0$ for all polytopes K, and, by continuity, for all $K \in \mathcal{K}(\mathbb{H}^3)$. $\square$

Open Problem. Theorems 5.3 and 5.5 are stated only for odd-dimensional hyperbolic spaces. The unsolved case for even-dimensional hyperbolic spaces is a gap that hinders the generalization of Corollary 5.6 to dimension 4 or greater. This limitation stems from Theorem 5.1, which has only been proven for even-dimensional spherical spaces. Generalization of Theorem 5.1 to odd-dimensional spheres remains an important open problem, both for its potential application to valuation characterizations in $\mathbb{H}^n$ (as outlined above), as well as for characterizing valuations on polytopes in the sphere, with related applications to the computation of kinematic integral formulas, angle-sum functionals on Euclidean polytopes, and other aspects of convex and integral geometry.

References

[Al1] S. Alesker. Continuous rotation invariant valuations on convex sets. *Ann. of Math.*, 149:977–1005, 1999.

[Al2] S. Alesker. On P. McMullen's conjecture on translation invariant valuations. *Adv. Math.*, 155:239–263, 2000.

[Al3] S. Alesker. Description of translation invariant valuations on convex sets with solution of P. McMullen's conjecture. *Geom. Funct. Anal.*, 11:244–272, 2001.

[An] J. Anderson. *Hyperbolic Geometry*. Springer-Verlag, New York, 1999.

[Bo] V. Boltianskii. *Hilbert's Third Problem*. Wiley, New York, 1979.

[Fu] J. H. G. Fu. Kinematic formulas in integral geometry. *Indiana Univ. Math. J.*, 39:1115–1154, 1990.

[Ga] R. Gardner. *Geometric Tomography*. Cambridge University Press, New York, 1995.

[GHS1] F. Gao, D. Hug, and R. Schneider. Intrinsic volumes and polar sets in spherical space. *Math. Notae*, 41:159–176, 2003.

[GHS2] J. Gates, D. Hug, and R. Schneider. Valuations on convex sets of oriented hyperplanes. *Discrete Comput. Geom.*, 33:57–65, 2005.

[Gr] H. Groemer. On the extension of additive functionals on classes of convex sets. *Pacific J. Math.*, 75:397–410, 1978.

[Ha] H. Hadwiger. *Vorlesungen über Inhalt, Oberfläche, und Isoperimetrie*. Springer-Verlag, Berlin, 1957.

[Ho] R. Howard. The kinematic formula in Riemannian homogeneous spaces. *Mem. Amer. Math. Soc.*, 106(509), 1993.

[Kl1] D. Klain. A short proof of Hadwiger's characterization theorem. *Mathematika*, 42:329–339, 1995.

[Kl2] D. Klain. Even valuations on convex bodies. *Trans. Amer. Math. Soc.*, 352:71–93, 2000.

[Kl3] D. Klain. Bonnesen-type inequalities for surfaces of constant curvature. Preprint, 2005. See also `http://faculty.uml.edu/dklain/Klain-Bonnesen.pdf`.

[KR] D. Klain and G.-C. Rota. *Introduction to Geometric Probability*. Cambridge University Press, New York, 1997.

[LR] M. Ludwig and M. Reitzner. A characterization of affine surface area. *Adv. Math.*, 147:138–172, 1999.

[Lud] M. Ludwig. A characterization of affine length and asymptotic approximation of convex discs. *Abh. Math. Sem. Univ. Hamburg*, 69:75–88, 1999.

[Lut1] E. Lutwak. Extended affine surface area. *Adv. Math.*, 85:39–68, 1991.

[Lut2] E. Lutwak. Selected affine isoperimetric inequalities. In P. Gruber and J. M. Wills, editors, *Handbook of Convex Geometry*, pages 151–176. North-Holland, Amsterdam, 1993.

[McM1] P. McMullen. Valuations and Euler-type relations on certain classes of convex polytopes. *Proc. London Math. Soc.*, 35:113–135, 1977.

[McM2] P. McMullen. The polytope algebra. *Adv. Math.*, 78:76–130, 1989.

[McM3] P. McMullen. Valuations and dissections. In P. Gruber and J. M. Wills, editors, *Handbook of Convex Geometry*, pages 933–988. North-Holland, Amsterdam, 1993.

[MS] P. McMullen and R. Schneider. Valuations on convex bodies. In P. Gruber and J. M. Wills, editors, *Convexity and Its Applications*, pages 170–247. Birkhäuser, Boston, MA, 1983.

[Mu] J. Munkres. *Elements of Algebraic Topology*. Benjamin/Cummings, Menlo Park, CA, 1984.

[Os] R. Osserman. Bonnesen-style isoperimetric inequalities. *Amer. Math. Monthly*, 86:1–29, 1979.

[Ru] W. Rudin. *Real and Complex Analysis*. McGraw-Hill, New York, 1987.

[Sah] C.-H. Sah. *Hilbert's Third Problem: Scissors Congruence*. Fearon Pitman, San Francisco, CA, 1979.

[San] L. A. Santaló. *Integral Geometry and Geometric Probability*. Addison-Wesley, Reading, MA, 1976.

[Sc1] R. Schneider. *Convex Bodies: The Brunn–Minkowski Theory*. Cambridge University Press, New York, 1993.

[Sc2] R. Schneider. Simple valuations on convex bodies. *Mathematika*, 43:32–39, 1996.

[St] J. Stillwell. *Geometry of Surfaces*. Springer-Verlag, New York, 1992.

[SW] R. Schneider and J. Wieacker. Integral geometry. In P. Gruber and J. M. Wills, editors, *Handbook of Convex Geometry*, pages 1349–1390. North-Holland, Amsterdam, 1993.

Received November 16, 2005, *and in revised form February* 21, 2006. *Online publication August* 4, 2006.