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The Minkowski problem for polytopes

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Abstract

The traditional solution to the Minkowski problem for polytopes involves two steps. First, the existence of a polytope satisfying given boundary data is demonstrated. In the second step, the uniqueness of that polytope (up to translation) is then shown to follow from the equality conditions of Minkowski's inequality, a generalized isoperimetric inequality for mixed volumes that is typically proved in a separate context. In this article we adapt the classical argument to prove both the existence theorem of Minkowski *and* his mixed volume inequality simultaneously, thereby providing a new proof of Minkowski's inequality that demonstrates the equiprimordial relationship between these two fundamental theorems of convex geometry.

Keywords: Polytope; Isoperimetric; Geometric inequality; Minkowski problem; Mixed volume; Brunn–Minkowski; Convex

It is easy to see that a convex polygon in $\mathbb{R}^2$ is uniquely determined (up to translation) by the directions and lengths of its edges. This suggests the following, less easily answered, question in higher dimensions: given a collection of proposed facet normals and facet areas, is there a convex polytope in $\mathbb{R}^n$ whose facets fit the given data, and, if so, is the resulting polytope unique? This question, along with its answer, is known as the *Minkowski problem*.

Denote by $\mathscr{P}^n$ the set of convex polytopes in $\mathbb{R}^n$. For $P \in \mathscr{P}^n$ and a unit vector $u \in \mathbb{R}^n$, denote by P_u the orthogonal projection of P onto the hyperplane $u^{\perp}$, and denote by P^u the support set of P in the direction of u. Since P is a polytope, P^u will be a face of P. If $\dim P = n$ and P^u is a face of dimension $n-1$ then P^u is called a

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facet of P, where u is the corresponding *facet normal*. The volume of a polytope P will be denoted by $V(P)$. If P is a subset of a hyperplane in $\mathbb{R}^n$, denote the $(n-1)$-dimensional volume of P by $v(P)$.

The *Minkowski problem for polytopes* concerns the following specific question: Given a collection $u_1, \ldots, u_k$ of unit vectors and $\alpha_1, \ldots, \alpha_k > 0$, under what condition does there exist a polytope P having the u_i as its facet normals and the α_i as its facet areas; that is, such that $v(P^{u_i}) = \alpha_i$ for each i?

A necessary condition on the facet normals and facet areas is given by the following proposition [BF48,Sch93a].

Proposition 1. *Suppose that $P \in \mathscr{P}^n$ has facet normals $u_1, u_2, \ldots, u_k$ and corresponding facet areas $\alpha_1, \alpha_2, \ldots, \alpha_k$. Then*

$$\alpha_1 u_1 + \cdots + \alpha_k u_k = 0. \tag{1}$$

Proof. If $u \in \mathbb{R}^n$ is a unit vector, then $|u_i \cdot u|\alpha_i$ is equal to the area of the orthogonal projection of the ith facet of P onto the hyperplane $u^\perp$. Summing over all facets whose outward normals form an acute angle with u, we obtain

$$\sum_{u_i \cdot u > 0} (u_i \cdot u)\alpha_i = v(P_u).$$

Meanwhile summing analogously over all facets whose outward normals form an obtuse angle with u yields the value $-v(P_u)$.

Let $w = \alpha_1 u_1 + \cdots + \alpha_k u_k$. It now follows that

$$w \cdot u = \sum_i (u_i \cdot u)\alpha_i = \sum_{u_i \cdot u > 0} (u_i \cdot u)\alpha_i + \sum_{u_i \cdot u < 0} (u_i \cdot u)\alpha_i = v(P_u) - v(P_u) = 0.$$

In other words, $w \cdot u = 0$ for all u, so that $w = 0$. $\square$

Proposition 1 illustrates a necessary condition for existence of a polytope having a given set of facet normals and facet areas. The remarkable discovery of Minkowski was that the converse of Proposition 1 (along with some minor additional assumptions) is also true. In other words, condition (1) is both necessary and (almost) sufficient, and, moreover, determines a polytope that is unique up to translation. To be more precise, we have the following theorem.

Theorem 2 (Solution to the Minkowski problem). *Suppose $u_1, u_2, \ldots, u_k \in \mathbb{R}^n$ are unit vectors that span $\mathbb{R}^n$, and suppose that $\alpha_1, \alpha_2, \ldots, \alpha_k > 0$. Then there exists a polytope $P \in \mathscr{P}^n$, having facet unit normals $u_1, u_2, \ldots, u_k$ and corresponding facet areas $\alpha_1, \alpha_2, \ldots, \alpha_k$, if and only if*

$$\alpha_1 u_1 + \cdots + \alpha_k u_k = 0.$$

Moreover, this polytope is unique up to translation.

Proofs of this theorem (and its many generalizations) abound in the literature. See, for example, any of [BF48,Lut93,Sch93a]. Once the surface data are suitably defined, the Minkowski problem can also be generalized to the context of compact convex sets [Sch93a], to the p-mixed volumes of the Brunn–Minkowski–Firey theory [Lut93], and to electrostatic capacity [Jer96]. See also [Sch93b] for a extensive survey of the Minkowski problem and its applications.

Minkowski's original proof of Theorem 2 involves two steps. First, the existence of a polytope satisfying the given facet data is demonstrated by a linear optimization argument. In the second step, the uniqueness of that polytope (up to translation) is then shown to follow from the equality conditions of *Minkowski's inequality* [BF48,Sch93a], a generalized isoperimetric inequality for *mixed volumes* that is typically proved in a separate context. In this article we show instead that the Minkowski problem and Minkowski's inequality are two facets of the same coin.

In the sections that follow, we prove Theorem 2 *without* assuming Minkowski's inequality. Instead, we prove both the existence theorem of Minkowski *and* his mixed volume inequality simultaneously, thereby demonstrating the fundamental and equiprimordial relationship between these two results. More specifically, we use the near-triviality of Theorem 2 in dimension 2 to prove Minkowski's inequality (for $\mathbb{R}^2$), and then bootstrap our way to proofs of both Theorem 2 and Minkowski's inequality in higher dimension, using dimension 2 as a basis for an induction argument on dimension. In the process it will be seen that the existence of a solution to Minkowski's problem corresponds to the weak version of Minkowski's inequality (without conditions for equality), while the uniqueness of that solution corresponds to the equality conditions of Minkowski's inequality and its equivalent formulation as the Brunn–Minkowski inequality.

1. Mixed volumes

A compact convex set K in $\mathbb{R}^n$ is determined uniquely by its *support function* $h_K : \mathbb{R}^n \to \mathbb{R}$, defined by $h_K(u) = \max_{x \in K}\{x \cdot u\}$, where $\cdot$ denotes the standard inner product on $\mathbb{R}^n$. For $K_1, K_2, \ldots, K_m \in \mathscr{P}^n$, and real numbers $\lambda_1, \lambda_2, \ldots, \lambda_m > 0$, define the *Minkowski linear combination* by

$$\lambda_1 K_1 + \lambda_2 K_2 + \cdots + \lambda_m K_m = \{\lambda_1 x_1 + \cdots + \lambda_m x_m \mid x_i \in K_i\}.$$

The support function of a Minkowski combination is given by

$$h_{\lambda_1 K_1 + \lambda_2 K_2 + \cdots + \lambda_m K_m} = \sum_{j=1}^{m} \lambda_j h_{K_j}.$$

It is not difficult to show that if the convex sets K_i are polytopes then the Minkowski combination is also a polytope.

The support function of a single point $x \in \mathbb{R}^n$ is given by the inner product: $h_x(u) = x \cdot u$. If a compact convex set K is translated by a vector x we have $h_{K+x}(u) = h_K(u) + x \cdot u$.

The support function can be used to compute the volume of a polytope, as described by the following proposition.

Proposition 3. *Suppose that $P \in \mathscr{P}^n$ has facet unit normals $u_1, u_2, \ldots, u_k$ and corresponding facet areas $v(P^{u_1}), v(P^{u_2}), \ldots, v(P^{u_k})$. Then the volume $V(P)$ is given by*

$$V(P) = \frac{1}{n} \sum_{i=1}^{k} h_P(u_i) v(P^{u_i}). \tag{2}$$

Proof. If P contains the origin, then (2) follows by summing over the volumes of each cone having base at a facet P^{u_i} of P, apex at the origin, and corresponding height $h_P(u_i)$. If P does not contain the origin, then translate P so that it does. Clearly the volume $V(P)$ is invariant under translation, while the translation invariance of the sum on the right-hand side of (2) follows from Proposition 1. □

The basis of the theory of mixed volumes is the polylinearization of volume with respect to Minkowski linear combinations: If $P_1, \ldots, P_m \in \mathscr{P}^n$ and $\lambda_1, \ldots, \lambda_m > 0$, then Euclidean volume V is a homogeneous polynomial in the positive variables $\lambda_1, \ldots, \lambda_m$; that is,

$$V(\lambda_1 P_1 + \cdots + \lambda_m P_m) = \sum_{i_1, \ldots, i_n = 1}^{m} V(P_{i_1}, \ldots, P_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n}, \tag{3}$$

where each symmetric coefficient $V(P_{i_1}, \ldots, P_{i_n})$ depends only on the bodies $P_{i_1}, \ldots, P_{i_n}$.

Given $P_1, \ldots, P_n \in \mathscr{P}^n$, the coefficient $V(P_1, \ldots, P_n)$ is called the *mixed volume* of the convex polytopes $P_1, \ldots, P_n$. It is well-known, but not trivial, that the mixed volume $V(P_1, \ldots, P_n)$ is a non-negative continuous function in n variables on the set $\mathscr{P}^n$, symmetric in the variables P_i, and monotonic with respect to the subset partial ordering on $\mathscr{P}^n$. If the P_i are all translates of a polytope P, then the mixed volume of the P_i is equal to the volume of P. In particular, $V(P, \ldots, P) = V(P)$.

The mixed volume is also linear in each parameter with respect to Minkowski sums; that is,

$$V(aP + bQ, P_2, \ldots, P_n) = aV(P, P_2, \ldots, P_n) + bV(Q, P_2, \ldots, P_n),$$

for all $a, b \geqslant 0$. Moreover, the translation invariance of volume in $\mathbb{R}^n$ implies that, if w is a single point in $\mathbb{R}^n$, then $V(P_1, \ldots, P_{n-1}, w) = 0$, and $V(P_1, \ldots, P_{n-1}, P_n + w) = V(P_1, \ldots, P_n)$ for all polytopes P_i.

For the special case of two convex polytopes P and Q, denote

$$V_i(P,Q) = V(\underbrace{P,\dots,P}_{n-i},\underbrace{Q,\dots,Q}_{i}).$$

For this case, Eq. (3) is known as *Steiner's formula*, specifically,

$$V(\lambda P + \mu Q) = \sum_{i=0}^{m} \binom{n}{i} V_i(P,Q)\lambda^{n-i}\mu^i \tag{4}$$

for all $\lambda, \mu \geqslant 0$. Note that in general $V_i(P,Q) \neq V_i(Q,P)$ (except where the dimension $n=2$ or when n is even and $i=n/2$). However, the symmetry of mixed volumes does imply that $V_i(P,Q) = V_{n-i}(Q,P)$. Moreover, $V_i(P,Q)$ is typically *not* linear in its parameters with respect to Minkowski sums for dimension $n>2$, with a notable exception:

$$V_1(P, aQ_1 + bQ_2) = aV_1(P,Q_1) + bV_1(P,Q_2),$$

for all $a,b \geqslant 0$. (That is, V_1 is linear in its second parameter only.)

From Steiner's formula (4), the volume formula (2), and elementary properties of polynomials, it can be shown that, for $i \geqslant 1$,

$$V_i(P,Q) = \frac{1}{n}\sum_u h_Q(u)v(\underbrace{P^u,\dots,P^u}_{n-i},\underbrace{Q^u,\dots,Q^u}_{i-1}), \tag{5}$$

where the sum is taken over all u such that $P+Q$ has a non-degenerate facet in the direction of u. In particular,

$$V_1(P,Q) = \frac{1}{n}\sum_u h_Q(u)v(P^u).$$

Mixed volumes are typically used to describe and measure the relationship between a compact convex set and its orthogonal projections onto subspaces. Such applications derive in part from the formula for the mixed volume of polytope P with a line segment $\overline{ox}$ having endpoints at the origin o and a point $x \in \mathbb{R}^n$. For $P \in \mathscr{P}^n$ and a non-zero vector $x \in \mathbb{R}^n$ denote by P_x the orthogonal projection of the set P onto the subspace $x^\perp$. By the Cavalieri principle for volume we have

$$V(P + \overline{ox}) = V(P) + |x|v(P_x),$$

for all $P \in \mathscr{P}^n$ and all vectors $x \neq 0$. It follows that all of the terms in Steiner's formula (4) vanish except two, that is,

$$V(P + \overline{ox}) = V(P) + nV(P,\dots,P,\overline{ox}) = V(P) + nV_1(P,\overline{ox}),$$

so that $nV_1(P, \overline{ox}) = |x|v(P_x)$. In particular, if $u \in \mathbb{R}^n$ is a *unit vector* then

$$nV_1(P, \overline{ou}) = v(P_u). \tag{6}$$

A detailed treatment of these and other properties of mixed volumes can be found in [BF48,Sch93a].

2. Existence

In this section we describe the solution to the existence part of the Minkowski problem. The proof of the Minkowski Existence Theorem presented in this section is essentially the original proof by Minkowski [BF48]. Constructions in this proof will then provide the tools necessary to describe a new proof of both the Uniqueness Theorem and the Brunn–Minkowski inequality in the sections that follow.

Theorem 4 (Minkowski Existence Theorem). *Suppose $u_1, u_2, \ldots, u_k \in \mathbb{R}^n$ are unit vectors that do not all lie in a hyperplane, and suppose that $\alpha_1, \alpha_2, \ldots, \alpha_k > 0$. If*

$$\alpha_1 u_1 + \cdots + \alpha_k u_k = 0, \tag{7}$$

then there exists a polytope $P \in \mathscr{P}^n$ having facet unit normals $u_1, u_2, \ldots, u_k$ and corresponding facet areas $\alpha_1, \alpha_2, \ldots, \alpha_k$.

Note that the spanning assumption on the vectors u_i in Theorem 4 implies that $k \geqslant n+1$.

Proof. For all $h = (h_1, \ldots, h_k) \in \mathbb{R}^k$, let H be the corresponding region of $\mathbb{R}^n$ given by

$$H = \{x \in \mathbb{R}^n \mid x \cdot u_i \leqslant h_i\}.$$

Evidently H is a closed convex region whose facet unit normals, if any, will form a subset of $\{u_1, \ldots, u_k\}$. Condition (7), along with the spanning condition on the u_i, guarantees that H will be bounded, that is, H is a (possibly empty) polytope. For each $i = 1, \ldots, k$, denote by β_i the area of the facet of H having normal u_i. If there is no such facet, set $\beta_i = 0$. Note that

$$V(H) = \frac{1}{n} \sum_{i=0}^{k} h_i \beta_i.$$

Define a map $\Phi : \mathbb{R}^k \to \mathbb{R}$ by

$$\Phi(h) = \Phi(h_1, \ldots, h_k) = \frac{1}{n} \sum_{i=1}^{k} h_i \alpha_i.$$

Condition (7) implies that Φ is translation invariant in the following sense: For all $w \in \mathbb{R}^n$,

$$\Phi(h_1 + w \cdot u_1, \ldots, h_k + w \cdot u_k) = \Phi(h). \tag{8}$$

In order to find the desired polytope P we will first show that Φ can be *minimized*, subject to the constraint $V(H) \geqslant 1$. The minimizing vector h for Φ will then provide the polytope we seek (after suitable scaling).

To this end, denote

$$\mathscr{H} = \{h \mid h_i \geqslant 0\} \cap \{V(H) \geqslant 1\}.$$

Note that $\mathscr{H}$ lies in the positive orthant of $\mathbb{R}^n$. Since Φ is a positive linear functional and the volume V is continuous, it follows that $\mathscr{H}$ is closed, so that Φ must attain a minimum $m = \Phi(h^*)$ for some $h^* \in \mathscr{H}$. Let H^* denote the polytope associated to the minimizing vector h^*. Since $h^* \in \mathscr{H}$, it follows that $V(H^*) \geqslant 1$ so that some $h_i > 0$. Hence, $m > 0$.

If $V(H) \geqslant 1$, then there exists w such that $H + w$ contains the origin in its interior, so that $\Phi(h) = \Phi(h + (w \cdot u_i)_{i=1}^k) \geqslant \Phi(h^*) = m$ by (8). In other words,

$$m = \min\{\Phi(h) \mid V(H) \geqslant 1\}.$$

Suppose $V(H^*) = \varepsilon > 1$. In this case $\Phi(\varepsilon^{-\frac{1}{n}} h^*) = \varepsilon^{-\frac{1}{n}} \Phi(h^*) < \Phi(h^*)$, while $V(\varepsilon^{-\frac{1}{n}} H^*) = 1$, contradicting the minimality of $\Phi(h^*)$. It follows that

$$V(H^*) = 1.$$

Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\beta = (\beta_1^*, \ldots, \beta_k^*)$, where the values of β_i^* are derived from the facets of H^*. We will show that

$$\frac{1}{n} \alpha \cdot X = m, \tag{9}$$

$$\frac{1}{n} \beta \cdot X = 1 \tag{10}$$

are equations for the *same* hyperplane, so that the vectors α and β are *parallel*, with $\beta = \frac{1}{m}\alpha$. This will imply that H^* satisfies the requirements of the theorem after suitable scaling.

Since $\Phi(h^*) = m$ and $V(H^*) = 1$, the point h^* lies in both hyperplanes, that is, the hyperplanes (9) and (10) must indeed intersect.

Next, suppose there is another point h^{**} in (9) with corresponding polytope H^{**}; that is,

$$m = \Phi(h^{**}) = \frac{1}{n} \sum_{i=0}^{k} h_i^{**} \alpha_i.$$

The minimality condition on m implies that $V(H^{**}) \leqslant 1$. For $0 \leqslant \lambda \leqslant 1$, let H_λ denote the polytope corresponding to the vector $\lambda h^* + (1-\lambda)h^{**}$. The linearity of Φ now implies that

$$\Phi(\lambda h^* + (1-\lambda)h^{**}) = m,$$

and the minimality condition on m again implies that $V(H_\lambda) \leqslant 1$. But

$$\lambda H^* + (1-\lambda)H^{**} \subseteq H_\lambda,$$

so that

$$V(\lambda H^* + (1-\lambda)H^{**}) \leqslant V(H_\lambda) \leqslant 1 = V(H^*)$$

for all $0 \leqslant \lambda \leqslant 1$. On expanding Steiner's formula (4) for $V(\lambda H^* + (1-\lambda)H^{**})$ we have

$$\sum_{i=0}^{n} V_{n-i}(H^*, H^{**}) \binom{n}{i} \lambda^i (1-\lambda)^{n-i} = V(\lambda H^* + (1-\lambda)H^{**}) \leqslant 1, \tag{11}$$

for all $0 \leqslant \lambda \leqslant 1$. On taking derivatives of the left-hand side term of (11) at $\lambda = 1$, we obtain

$$1 \geqslant V_1(H^*, H^{**}) = V(H^*, \ldots, H^*, H^{**}) = \frac{1}{n} \sum_{i=0}^{k} h_i^{**} \beta_i.$$

It follows that h^{**} lies in one half-space bounded by (10). That is, the entire hyperplane (9) lies in one half-space bounded by (10). This can only occur if the two hyperplanes are parallel. Since they intersect, the two hyperplanes must be identical, with $\beta = \frac{1}{m}\alpha$.

Now let

$$P = m^{\frac{1}{n-1}} H^*.$$

The polytopes P and H^* share the same facet normals u_i, while $v(P^{u_i}) = (m^{\frac{1}{n-1}})^{n-1}\beta_i = m\beta_i = \alpha_i$, as required. $\square$

In the previous argument we obtained a polytope $P = m^{\frac{1}{n-1}} H^*$ such that $V(H^*) = 1$. It follows that

$$V(P) = m^{\frac{n}{n-1}} V(H^*) = m^{\frac{n}{n-1}}.$$

Moreover, for all h such that $V(H) = 1$, we have shown that

$$V_1(P, H) = \frac{1}{n} \sum_i h_i \alpha_i = \Phi(h) \geqslant \Phi(h^*) = m = V(P)^{\frac{n-1}{n}} V(H)^{\frac{1}{n}}.$$

Hence, if Q is a polytope in $\mathbb{R}^n$ such that $V(Q) = 1$ *and* the facet normals of Q form a subset of the facet normals of P

$$V_1(P, Q) \geqslant V(P)^{\frac{n-1}{n}} V(Q)^{\frac{1}{n}}. \tag{12}$$

From the homogeneity of V_1 and V it follows that (12) holds regardless of the value of $V(Q)$. Note, however, that (12) has still only been demonstrated for the special case in which P is a minimizer for the linear functional Φ and the facet normals of Q form a subset of the facet normals of P.

We will generalize inequality (12) further in the sections that follow.

3. Uniqueness in $\mathbb{R}^2$

Volume in $\mathbb{R}^2$ is commonly referred to as *area*, and we will denote by $A(K)$ the area of a region $K \subseteq \mathbb{R}^2$. In this instance the mixed volume formula (4) for the volume (area) of a Minkowski sum becomes

$$A(\lambda K + \mu L) = A(K)\lambda^2 + 2A(K, L)\lambda\mu + A(L)\mu^2, \tag{13}$$

for all $\lambda, \mu \geqslant 0$. Note also that $V_1(K, L) = A(K, L)$. Since mixed volumes are symmetric in their entries, $A(K, L) = A(L, K)$, and so $V_1(K, L) = V_1(L, K)$ for all $K, L \in \mathscr{P}^2$. Recall that this remarkable symmetry does not hold in higher dimension.

The solution to the Minkowski problem in $\mathbb{R}^2$ turns out to be almost trivial to derive. Given a collection of unit normals u_i and corresponding edge lengths $\alpha_i > 0$, we can rotate the edge normals counter-clockwise by 90° and lengthen each by its given edge length, transforming the normals into *actual edges*. Assuming we have listed the normals in *counter-clockwise order* (to prevent "looping"), we can just lay these oriented edges end-to-end and construct the polygonal closed curve that describes the corresponding polygon in a unique way up to translation; i.e. depending only on where we set down the pen to draw the first edge.

The details, which require some bookkeeping, are described as follows.

Theorem 5 (Existence theorem for polygons). *Suppose $u_1, u_2, \ldots, u_k \in \mathbb{R}^2$ are unit vectors that span $\mathbb{R}^2$, and suppose that $\alpha_1, \alpha_2, \ldots, \alpha_k > 0$. There exists a polygon $P \in \mathscr{P}^2$ having edge unit normals $u_1, u_2, \ldots, u_k$, and corresponding edge lengths $\alpha_1, \alpha_2, \ldots, \alpha_k$, if and only if*

$$\alpha_1 u_1 + \cdots + \alpha_k u_k = 0. \tag{14}$$

Moreover, such a polygon P is unique up to translation.

Note that the spanning assumption on the vectors u_i in Theorem 4 implies that $k \geqslant 3$.

Proof of Theorem 5. Suppose that a polygon P has boundary data given by the normals u_i and edge-lengths α_i. Let ϕ denote the counter-clockwise rotation of $\mathbb{R}^2$ by the angle $\pi/2$. For each i let $v_i = \phi(\alpha_i u_i)$. Then each v_i is congruent by a *translation* to the ith edge of the polygon P. Since the boundary of a convex polygon is a simple closed curve, we have

$$v_1 + \cdots + v_k = 0.$$

On applying ϕ^{-1} to this identity, we obtain (14).

Conversely, suppose that a family of unit vectors u_i and positive real numbers α_i satisfy (14), where the vectors u_i span $\mathbb{R}^2$. As above, let $v_i = \phi(\alpha_i u_i)$ for each i. Assume also that the vectors u_i (and therefore, the vectors v_i) are indexed in counter-clockwise order around the circle. We will construct a polygon P having boundary data given by the normals u_i and edge-lengths α_i. Condition (14) implies that $v_1 + \cdots + v_k = 0$.

Denote

$$\begin{aligned} x_1 &= v_1 \\ x_2 &= v_1 + v_2 \\ &\vdots \\ x_k &= v_1 + \cdots + v_k = o \end{aligned}$$

and let P denote the convex hull of the points $x_1, x_2, \ldots, x_k$ (where $x_k = o$, the origin). We will show that each x_i is an extreme point of P. It will then follow that the x_i are the vertices of P, so that the edges of P are congruent to the vectors v_i, as required.

To show that each x_i is an extreme point, it is sufficient to consider the case of $x_k = o$. Moreover, since convex dependence relations are invariant under rigid motions, we may assume without loss of generality that v_1 points along the positive x-axis. Let $\hat{\jmath} = (0, 1)$. Note that $x_1 \cdot \hat{\jmath} = v_1 \cdot \hat{\jmath} = 0$, and that if $v_s \cdot \hat{\jmath} < 0$ and $s \leqslant t \leqslant k$ then $v_t \cdot \hat{\jmath} < 0$, since the v_i are arranged in counter-clockwise order. Moreover, since $\sum v_i = o$ and since the v_i span $\mathbb{R}^2$, we must have $v_2 \cdot \hat{\jmath} > 0$.

Now suppose that o is *not* an extreme point of P. In this case, $o = a_1 x_1 + \cdots + a_k x_k$, where each $a_i \geqslant 0$ and $a_1 + \cdots + a_k = 1$. Note that

$$0 = o \cdot \hat{\jmath} = \sum_i a_i (x_i \cdot \hat{\jmath}). \tag{15}$$

Since $x_2 \cdot \hat{\jmath} > 0$, it follows from (15) that some $x_s \cdot \hat{\jmath} < 0$. Because $v_i \cdot \hat{\jmath} < 0$ for all $i \geqslant s$, we have

$$0 = o \cdot \hat{\jmath} = x_k \cdot \hat{\jmath} = \left(x_s + \sum_{i > s} v_i \right) \cdot \hat{\jmath} < 0,$$

a contradiction. It follows that o (and similarly each other x_i) must be an extreme point of P. It also follows that $x_s \cdot \hat{\jmath} \geqslant 0$ for all s, so that v_1 (and similarly each other v_i) must be parallel to edges of P.

Since we have given an explicit reconstruction of the boundary of P from the normals u_i and edge lengths α_i, starting from a base point—in this case the origin o—it also follows that such a polygon P is unique up to the choice of that base point, in other words, up to translation. □

Equality conditions for geometric inequalities frequently involve the equivalence relation of *homothesis*. Two subsets P and Q of $\mathbb{R}^n$ are said to be *homothetic* if there exist $a > 0$ and $x \in \mathbb{R}^n$ such that $P = aQ + x$. In other words, homothetic sets differ only by translations and dilations.

Corollary 6 (Minkowski's inequality in $\mathbb{R}^2$). *Suppose that P and Q are polygons in $\mathbb{R}^2$. Then*

$$A(P, Q)^2 \geqslant A(P)A(Q).$$

If P and Q have non-empty interiors, then equality holds if and only if P and Q are homothetic.

Proof. If $A(P) = 0$ or $A(Q) = 0$ then the inequality is trivial.

Suppose that both P and Q have non-empty interiors. From the quadratic homogeneity of area we may assume without loss of generality that $A(P) = A(Q) = 1$. It then suffices to show that $A(P, Q) \geqslant 1$, with equality if and only if P and Q are translates.

Let $\alpha_1, \ldots, \alpha_k$ denote the edge lengths for the polygon P, where the corresponding edge normals for P are $u_1, \ldots, u_k$. Theorem 5 implies that, up to translation, P is indeed the (unique) polygon that provides a minimizer for the functional Φ in the proof of Theorem 4, where Φ is now defined using the edge normals and lengths for the polygon P. It follows that inequality (12) applies to the polygon P.

Note that we cannot immediately apply (12) to $A(P, Q)$, because the vectors u_i may not be edge unit normals for Q. Instead, let

$$\hat{Q} = \bigcap_i \{x \in \mathbb{R}^n \mid x \cdot u_i \leqslant h_Q(u_i)\}.$$

Evidently $Q \subseteq \hat{Q}$, so that $A(\hat{Q}) \geqslant A(Q) = 1$. Meanwhile $h_{\hat{Q}}(u_i) \leqslant h_Q(u_i)$ for all i, by the definition of $\hat{Q}$. Since the facet normals of $\hat{Q}$ do indeed form a subset of the facet normals of P, it follows from (12) that $A(P, \hat{Q}) = V_1(P, \hat{Q}) \geqslant 1$. Hence,

$$A(P, Q) = V_1(P, Q) = \sum_i h_Q(u_i)\alpha_i \geqslant \sum_i h_{\hat{Q}}(u_i)\alpha_i = V_1(P, \hat{Q}) = A(P, \hat{Q}) \geqslant 1.$$

For the equality case, suppose that $A(P,Q) = 1 = A(P,P)$. This implies that $A(P,\hat{Q}) = 1$ as well, so that *both* P and $\hat{Q}$ provide minimizers for the function Φ in the proof of Theorem 4. It follows that P and $\hat{Q}$ have the same facet data α. Theorem 5 then implies that $P = \hat{Q}$ up to translation, so that $A(\hat{Q}) = 1 = A(Q)$. Since $Q \subseteq \hat{Q}$, it follows that $\hat{Q} = Q$, so that $Q = P$ up to translation. $\square$

Minkowski's inequality (Corollary 6) is perhaps better known through its equivalent formulation, which describes area as a concave function with respect to Minkowski combinations.

Corollary 7 (Brunn–Minkowski inequality in $\mathbb{R}^2$). *Suppose that P and Q are polygons in $\mathbb{R}^2$. Then for $0 \leqslant \lambda \leqslant 1$,*

$$A((1-\lambda)P + \lambda Q)^{1/2} \geqslant (1-\lambda)A(P)^{1/2} + \lambda A(Q)^{1/2}.$$

If P and Q have non-empty interiors, then equality holds if and only if P and Q are homothetic.

Proof. If $A(P) = 0$ or $A(Q) = 0$ then the inequality is trivial.

Suppose that P and Q have non-empty interiors. Combining Steiner's formula for area (13) with Minkowski's inequality (Corollary 6) yields

$$\begin{aligned} A((1-\lambda)P + \lambda Q) &= (1-\lambda)^2 A(P) + 2\lambda(1-\lambda)A(P,Q) + \lambda^2 A(Q) \\ &\geqslant (1-\lambda)^2 A(P) + 2\lambda(1-\lambda)A(P)^{1/2}A(Q)^{1/2} + \lambda^2 A(Q) \\ &= [(1-\lambda)A(P)^{1/2} + \lambda A(Q)^{1/2}]^2, \end{aligned}$$

so that

$$A((1-\lambda)P + \lambda Q)^{1/2} \geqslant (1-\lambda)A(P)^{1/2} + \lambda A(Q)^{1/2},$$

with equality conditions identical to those of Minkowski's inequality (Corollary 6). $\square$

4. Uniqueness in $\mathbb{R}^n$

The following lemma will contribute to the general case of $\mathbb{R}^n$, for $n \geqslant 3$. Recall that K_u denotes the orthogonal projection of a polytope K onto the hyperplane $u^\perp$.

Lemma 8 (Rogers' Lemma). *Suppose that K and L are convex polytopes in $\mathbb{R}^n$, where $n \geqslant 3$. Suppose also that, for each unit vector u, we have $K_u = L_u$ up to translation. Then there exists a vector x such that $K = L + x$. That is, $K = L$ up to translation.*

This lemma is a special case of a theorem of Rogers [Rog65].

Proof of Lemma 8. Translate K and L into the positive orthant of $\mathbb{R}^n$ so that each coordinate hyperplane is a supporting hyperplane for both K and L. From here it suffices to show that $K = L$.

Let $e_1, \ldots, e_n$ denote the standard basis for $\mathbb{R}^n$. The hypotheses of the lemma assert that $K_{e_1} = L_{e_1}$ up to translation. Since the coordinate hyperplanes of $\mathbb{R}^{n-1}$ support K_{e_1} and L_{e_1}, it follows that the required translation is trivial (i.e. the zero vector) and $K_{e_1} = L_{e_1}$ identically. Similarly $K_{e_i} = L_{e_i}$ for each $i = 1, \ldots, n$. It follows that $h_{K_{e_i}} = h_{L_{e_i}}$ for each i. Since the support function $h_{K_{e_i}}$ is obtained by restricting the support function h_K to the hyperplane $e_i^{\perp}$, it follows that $h_K(w) = h_L(w)$ for all $w \in e_1^{\perp} \cup \cdots \cup e_n^{\perp}$.

Suppose that $u \in \mathbb{R}^n$ is a unit vector that is not contained in any coordinate hyperplane $e_i^{\perp}$. Since $K_u = L_u$ up to translation, there exists $x \in u^{\perp}$ such that $h_{K_u}(w) = h_{L_u}(w) + x \cdot w$ for all $w \in u^{\perp}$.

But $h_{K_u}(w) = h_{K_{e_i}}(w) = h_{L_{e_i}}(w) = h_{L_u}(w)$ for all $w \in u^{\perp} \cap e_i^{\perp}$, so that $x \cdot w = 0$ for all $w \in u^{\perp} \cap e_i^{\perp}$. In other words, $x \in (u^{\perp} \cap e_i^{\perp})^{\perp} = \mathrm{Span}(u, e_i)$ for each i. The assumption that $u \notin e_3^{\perp}$ implies that $\mathrm{Span}(u, e_1) \neq \mathrm{Span}(u, e_2)$, so that

$$x \in \mathrm{Span}(u, e_1) \cap \mathrm{Span}(u, e_2) = \mathrm{Span}(u).$$

Because $x \in u^{\perp}$, it now follows that $x = 0$, and that

$$h_K(u) = h_{K_u}(w) = h_{L_u}(w) = h_L(u)$$

for all $w \in u^{\perp}$, provided u is not contained in any coordinate hyperplane. Because support functions are continuous, it then follows that $h_K = h_L$ and that $K = L$. $\square$

The proof of uniqueness for the solution to the Minkowski problem and the proofs of the Minkowski and Brunn–Minkowski inequalities for dimensions greater than two will proceed according to the following inductive scheme:

1. The Minkowski and Brunn–Minkowski inequalities in dimension $n-1$ will be used to prove the uniqueness of the solution to the Minkowski problem in dimension n.
2. The uniqueness of the solution to the Minkowski problem in dimension n will then imply Minkowski's inequality in dimension n.
3. Minkowski's inequality in dimension n will then imply the Brunn–Minkowski inequality in dimension n.

The base case for this inductive argument was treated in the previous section, where both the Minkowski problem and the (Brunn–)Minkowski inequalities were established for dimension 2.

Theorem 9 (Minkowski's Uniqueness Theorem). *Suppose that K and L are polytopes in $\mathbb{R}^n$, where $n \geqslant 3$, and suppose also that K and L solve the Minkowski problem for* *facet unit normals* $u_1, \ldots, u_k$ *and corresponding facet areas* $\alpha_1, \ldots, \alpha_k$. *Then* $K = L$ *up to translation.*

Proof. Theorem 4 asserts that there exists a polytope P that minimizes the functional $\Phi(h)$ (subject to the constraint $V(H) \geqslant 1$) and has the desired facet normals and facet areas. It is therefore sufficient to show that $K = P$ up to translation (so that the same would hold for L).

Since K and P share the same corresponding facet areas α_i,

$$V_1(K, Q) = \frac{1}{n} \sum_i h_Q(u_i)\alpha_i = V_1(P, Q),$$

for all polytopes Q. In particular,

$$V(K) = V_1(K, K) = V_1(P, K) \quad \text{and} \quad V(P) = V_1(P, P) = V_1(K, P).$$

It follows from Steiner's formula (4) that

$$\begin{aligned} V(K+P) &= V(K) + nV_1(K,P) + \left(\sum_{i=2}^{n-2} \binom{n}{i} V_i(K,P) \right) + nV_1(P,K) + V(P) \\ &= (n+1)V(K) + (n+1)V(P) + \sum_{i=2}^{n-2} \binom{n}{i} V_i(K,P). \end{aligned} \tag{16}$$

Note that, for each i, the facet $(K+P)^{u_i}$ of the polytope $K+P$ having unit normal u_i is given by

$$(K+P)^{u_i} = K^{u_i} + P^{u_i}.$$

Meanwhile,

$$\begin{aligned} V(K+P) &= V_1(K+P, K+P) = V_1(K+P, K) + V_1(K+P, P) \\ &= \frac{1}{n} \left(\sum_i h_K(u_i) v(K^{u_i} + P^{u_i}) + \sum_i h_P(u_i) v(K^{u_i} + P^{u_i}) \right), \end{aligned} \tag{17}$$

where v denotes $(n-1)$-dimensional volume.

Steiner's formula (4) applied in dimension $(n-1)$ asserts that

$$v(K^{u_i} + P^{u_i}) = \sum_{j=0}^{n-1} \binom{n-1}{j} v_j(K^{u_i}, P^{u_i}),$$

where we denote

$$v_j(K^{u_i}, P^{u_i}) = v(\underbrace{K^{u_i}, \ldots, K^{u_i}}_{n-j-1}, \underbrace{P^{u_i}, \ldots, P^{u_i}}_{j}),$$

the $(n-1)$-dimensional mixed volume, and where $v(K^{u_i},\dots,K^{u_i}) = v(K^{u_i}) = \alpha_i = v(P^{u_i}) = v(P^{u_i},\dots,P^{u_i})$. Minkowski's inequality in dimension $(n-1)$ then implies that

$$v(K^{u_i},\dots,K^{u_i},P^{u_i}) \geqslant v(K^{u_i})^{\frac{n-2}{n-1}} v(P^{u_i})^{\frac{1}{n-1}} = \alpha_i^{\frac{n-2}{n-1}} \alpha_i^{\frac{1}{n-1}} = \alpha_i$$

and similarly $v(P^{u_i},\dots,P^{u_i},K^{u_i}) \geqslant \alpha_i$, with equality if and only if the P^{u_i} and K^{u_i} are translates. Hence,

$$v(K^{u_i}+P^{u_i}) \geqslant (n+1)\alpha_i + \sum_{j=1}^{n-3} \binom{n-1}{j} v_j(K^{u_i},P^{u_i}) \tag{18}$$

and similarly

$$v(K^{u_i}+P^{u_i}) \geqslant (n+1)\alpha_i + \sum_{j=2}^{n-2} \binom{n-1}{j} v_j(K^{u_i},P^{u_i}) \tag{19}$$

with equality in either case if and only if the P^{u_i} and K^{u_i} are translates.

Combining (18) and (19) with (17) yields

$$\begin{aligned}
V(K+P) &= \frac{1}{n} \sum_u \left(h_K(u) v(K^u+P^u) + h_P(u) v(K^u+P^u)\right) \\
&\geqslant \frac{1}{n} \sum_i \left(h_K(u_i) v(K^{u_i}+P^{u_i}) + h_P(u_i) v(K^{u_i}+P^{u_i})\right) \\
&\geqslant \frac{1}{n} \sum_i h_K(u_i) \left((n+1)\alpha_i + \sum_{j=2}^{n-2} \binom{n-1}{j} v_j(K^{u_i},P^{u_i}) \right) \\
&\quad + \frac{1}{n} \sum_i h_P(u_i) \left((n+1)\alpha_i + \sum_{j=1}^{n-3} \binom{n-1}{j} v_j(K^{u_i},P^{u_i}) \right) \\
&= (n+1)V(K) + (n+1)V(P) + \sum_{j=2}^{n-2} \binom{n}{j} V_j(K,P).
\end{aligned} \tag{20}$$

The first inequality is due to the possibility that $K+P$ has facet normal directions u that did not appear for K (and P) separately, while the second inequality follows from (18) and (19). The last equation in (20) follows from (5) and the fact that $V_j(K,P) = V_{n-j}(P,K)$. Note that equality holds throughout sequence (20) of inequalities if and only if the facets P^{u_i} and K^{u_i} are translates for each u_i *and* the facet normals of $K+P$ are identical to the facet normals of K (and P). But (16) asserts that equality indeed holds! Hence, P^{u_i} and K^{u_i} are translates for each u_i, and $K+P$ contains no additional facet normals (not already accounted for by K and P).

We have shown that the polytope $\frac{1}{2}K + \frac{1}{2}P$ has facet normals $u_1, \dots, u_k$. Moreover, $\frac{1}{2}K + \frac{1}{2}P$ has facet areas $\alpha_1, \dots, \alpha_k$, because the corresponding facets of K and P in each facet normal direction are *translates*. This implies that

$$V_1((1/2)K + (1/2)P, Q) = V_1(K, Q) = V_1(L, Q),$$

for all polytopes Q. In particular, for any unit vector $u \in \mathbb{R}^n$,

$$V_1((1/2)K + (1/2)P, \overline{ou}) = V_1(K, \overline{ou}) = V_1(P, \overline{ou}).$$

It follows from (6) that

$$v((1/2)K_u + (1/2)P_u) = v(K_u) = v(P_u).$$

In other words, the orthogonal projections K_u and P_u satisfy the equality case of the Brunn–Minkowski inequality in $\mathbb{R}^{n-1}$. From the equality conditions of the Brunn–Minkowski inequality *in dimension* $n-1$ we have $K_u = P_u$ (up to translation) for all u. Since $\dim(P) = \dim(K) \geqslant 3$, we have $K = P$ (up to translation) by Rogers' Lemma 8. $\square$

Theorem 10 (Minkowski's inequality). *Suppose that P and Q are polytopes in $\mathbb{R}^n$. Then*

$$V_1(P, Q)^n \geqslant V(P)^{n-1} V(Q).$$

If P and Q have non-empty interiors, then equality holds if and only if P and Q are homothetic.

Proof. If $V(P) = 0$ or $V(Q) = 0$ then the inequality is trivial.

Suppose that both P and Q have non-empty interiors. Recall that n-dimensional volume is positively homogeneous of degree n; that is, for all $a, b > 0$ and all polytopes P, Q, we have $V(aP) = a^n V(P)$ and $V_1(aP, bQ) = a^{n-1} b V_1(P, Q)$. It is therefore sufficient to show that, if $V(P) = V(Q) = 1$, then $V_1(P, Q) \geqslant 1$, with equality iff P and Q are translates.

Let $\alpha_1, \dots, \alpha_k$ denote the facet areas of the polytope P, where the facet normals for P are $u_1, \dots, u_k$. The Uniqueness Theorem 9 implies that, up to translation, P is indeed the *unique* polytope that provides a minimizer for the functional Φ in the proof of Theorem 4, where Φ is defined using the facet normals and areas for the polytope P. It follows that inequality (12) applies to the polytope P.

We cannot immediately apply (12) to $V_1(P, Q)$, because the u_i may not be facet normals for Q. Instead, let

$$\hat{Q} = \bigcap_i \{x \in \mathbb{R}^n \mid x \cdot u_i \leqslant h_Q(u_i)\}.$$

Evidently $Q \subseteq \hat{Q}$, so that $V(\hat{Q}) \geqslant V(Q) = 1$. Meanwhile $h_{\hat{Q}}(u_i) \leqslant h_Q(u_i)$ for all i. Since the facet normals of $\hat{Q}$ do form a subset of the facet normals of P, it follows from (12) that $V_1(P, \hat{Q}) \geqslant V(P)^{\frac{n-1}{n}} V(\hat{Q})^{\frac{1}{n}} \geqslant 1$. Hence,

$$V_1(P, Q) = \sum_i h_Q(u_i)\alpha_i \geqslant \sum_i h_{\hat{Q}}(u_i)\alpha_i = V_1(P, \hat{Q}) \geqslant 1.$$

For the equality case, suppose that $V_1(P, Q) = 1 = V_1(P, P)$. This implies that $V_1(P, \hat{Q}) = 1$ as well. Since $V(\hat{Q}) \geqslant 1$, this implies that both P and $\hat{Q}$ provide minimizers for the function Φ in the proof of Theorem 4. It follows that P and $\hat{Q}$ have the same facet unit normals and corresponding facet areas. Theorem 9 then implies that $P = \hat{Q}$ up to translation, so that $V(\hat{Q}) = 1 = V(Q)$. Since $Q \subseteq \hat{Q}$, it follows that $\hat{Q} = Q$, so that $Q = P$ up to translation. $\square$

Theorem 11 (Brunn–Minkowski inequality). *Suppose that P and Q are polytopes in $\mathbb{R}^n$. For $0 \leqslant \lambda \leqslant 1$,*

$$V((1-\lambda)P + \lambda Q)^{1/n} \geqslant (1-\lambda)V(P)^{1/n} + \lambda V(Q)^{1/n}.$$

If P and Q have non-empty interiors, then equality holds if and only if P and Q are homothetic.

Proof. The inequality is trivial if $V(P) = 0$ or if $V(Q) = 0$.

Suppose that both P and Q have non-empty interiors. From the homogeneity of volume it is again sufficient to prove that $V((1-\lambda)P + \lambda Q) \geqslant 1$ when $V(P) = V(Q) = 1$, with equality if and only if P and Q are translates.

Recall that $V_1(P, P) = V(P)$ for all P. Suppose that $V(P) = V(Q) = 1$. From the linearity of the functional V_1 in its second parameter, we obtain

$$\begin{aligned} V((1-\lambda)P + \lambda Q) &= V_1((1-\lambda)P + \lambda Q, (1-\lambda)P + \lambda Q) \\ &= (1-\lambda)V_1((1-\lambda)P + \lambda Q, P) + \lambda V_1(1-\lambda)P + \lambda Q, Q) \\ &\geqslant (1-\lambda)V((1-\lambda)P + \lambda Q)^{\frac{n-1}{n}} V(P)^{\frac{1}{n}} \\ &\quad + \lambda V((1-\lambda)P + \lambda Q)^{\frac{n-1}{n}} V(Q)^{\frac{1}{n}} \\ &= V((1-\lambda)P + \lambda Q)^{\frac{n-1}{n}}, \end{aligned}$$

where the inequality follows from Minkowski's inequality (Theorem 10). It follows that

$$V((1-\lambda)P + \lambda Q) \geqslant 1,$$

where the equality conditions are identical to those of Minkowski's inequality (Theorem 10). $\square$

Remark. Although it is not needed here, we observe that the Brunn–Minkowski Theorem is easily seen to be *equivalent* to the Minkowski inequality. We have shown that the Minkowski inequality implies the Brunn–Minkowski Theorem. For the reverse implication, suppose that $V(P) = V(Q) = 1$. By Brunn–Minkowski, $V((1-\lambda)P + \lambda Q) \geqslant 1$ for all $0 \leqslant \lambda \leqslant 1$. It follows that the polynomial in λ obtained from the Steiner formula (4) for $V((1-\lambda)P + \lambda Q)$ is decreasing at $\lambda = 1$, so that its derivative at $\lambda = 1$ is non-positive. This in turn implies, after a simple calculation, that

$$V_1(P, Q) \geqslant 1.$$

This completes the network of implications founded by the two-dimensional case, and proceeding inductively by dimension as follows:

$$\begin{array}{ccc} \text{Minkowski's inequality} & \Leftrightarrow & \text{Brunn}-\text{Minkowski inequality} \\ \text{in dimension } (n-1) & & \text{in dimension } (n-1) \\ & \Downarrow & \\ & \text{Uniqueness Theorem} & \\ & \text{in dimension } n & \\ & \Downarrow & \\ & \text{Minkowski's inequality} & \\ & \text{in dimension } n & \end{array}$$

Since any convex body in $\mathbb{R}^n$ can be approximated by polytopes, the continuity of mixed volumes implies that the *inequalities* of Theorems 10 and 11 hold for compact convex sets as well as polytopes. However, this continuity argument does *not* demonstrate that the equality conditions hold in that more general context. It is true, however, that the equality conditions for the Brunn–Minkowski and Minkowski inequalities in the case of general convex bodies are the same as for polytopes (homothety) [Sch93a]. (For an approach similar to the one presented here, see [Kla02].) Moreover, these inequalities have inspired numerous applications [BZ88,Gar95,Gro96,LvGM96,LO95,San76,Sch93a,Tho96], generalizations [BZ88,BL76,Lut93,Sch93a], and analogues [Bor83,CJL96].

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