A Continuous Analogue of Sperner's Theorem

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Abstract

One of the best-known results of extremal combinatorics is Sperner's theorem, which asserts that the maximum size of an antichain of subsets of an n-element set equals the binomial coefficient $\binom{n}{\langle n/2\rangle}$, that is, the maximum of the binomial coefficients. In the last twenty years, Sperner's theorem has been generalized to wide classes of partially ordered sets.

It is the purpose of the present paper to propose yet another generalization that strikes in a different direction. We consider the lattice $\mathrm{Mod}(n)$ of linear subspaces (through the origin) of the vector space $\mathbb{R}^n$. Because this lattice is infinite, the usual methods of extremal set theory do not apply to it. It turns out, however, that the set of elements of rank k of the lattice $\mathrm{Mod}(n)$, that is, the set of all subspaces of dimension k of $\mathbb{R}^n$, or Grassmannian, possesses an invariant measure that is unique up to a multiplicative constant. Can this multiplicative constant be chosen in such a way that an analogue of Sperner's theorem holds for $\mathrm{Mod}(n)$, with measures on Grassmannians replacing binomial coefficients? We show that there is a way of choosing such constants for each level of the lattice $\mathrm{Mod}(n)$ that is natural and unique in the sense defined below and for which an analogue of Sperner's theorem can be proven.

The methods of the present note indicate that other results of extremal set theory may be generalized to the lattice $\mathrm{Mod}(n)$ by similar reasoning. ©1997 John Wiley & Sons, Inc.

1 The Lattice of Subspaces

Let $\mathrm{Mod}(n)$ denote the set of all linear subspaces of $\mathbb{R}^n$, that is, the set of all linear varieties passing through the origin (having fixed an origin once and for all). The set $\mathrm{Mod}(n)$ is a partially ordered set under the relation of inclusion of linear subspaces. Moreover, it is a lattice, where the join $x \vee y$ and the meet $x \wedge y$ of two elements $x, y \in \mathrm{Mod}(n)$ are defined, respectively, as the linear subspaces spanned by x and y and as the intersection of x and y. The lattice $\mathrm{Mod}(n)$ may be viewed as a continuous analogue of the lattice $P(S)$ of subsets of a set S with n elements. Note, however, that this analogy is only a partial one, since the distributive law governing unions and intersections of subsets of S does not hold in the lattice $\mathrm{Mod}(n)$. Nonetheless, this analogy shall carry us as far as we need to go.

We shall apply to the lattice $\mathrm{Mod}(n)$ the notions of *chain*, *flag*, and *antichain*, which are defined for any partially ordered set. An element x of

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$\mathrm{Mod}(n)$ has rank k (that is, $r(x) = k$) whenever x is a linear subspace of dimension k. The subspace $\{0\}$ is the minimal element of the lattice $\mathrm{Mod}(n)$.

Much as the group of permutations of the set S acts naturally on $P(S)$, the *orthogonal group* $O(n)$ acts naturally on the lattice $\mathrm{Mod}(n)$.

Recall that set of all elements of $\mathrm{Mod}(n)$ of dimension (rank) k, denoted $G(n,k)$, is called the *Grassmannian*. There exists an invariant (Haar) measure acting on $G(n,k)$ that is unique up to a common factor (see [15]). In the appropriate normalization, the total measure of $G(n,k)$ should be in some sense an analogue of the binomial coefficient.

Let τ_n denote the invariant measure on $G(n,1)$, that is, on the set of all straight lines through the origin. Denote

$$[n] = \tau_n(G(n,1))\,. \tag{1.1}$$

The value of $[n]$ depends on the normalization for the measure τ_n, to be determined later.

Let $\mathrm{Flag}(n)$ be the set of flags in $\mathrm{Mod}(n)$. For $x \in \mathrm{Mod}(n)$, denote by $\mathrm{Flag}(x)$ the set of all flags that contain x, that is, the set of all sequences $(x_0, x_1, \dots, x_n)$ of $x_i \in \mathrm{Mod}(n)$ where $\dim(x_i) = i$ such that $x_0 \leq x_1 \leq \cdots \leq x_n$ and such that one of the x_i equals x. Note that $x_0 = \{0\}$ and $x_n = \mathbb{R}^n$.

For fixed x_k, the set of all sequences $(x_k, x_{k+1}, \dots, x_n)$ with $x_i \in \mathrm{Mod}(n)$ such that $\dim(x_i) = i$ and $x_i \leq x_{i+1}$ is isomorphic to $\mathrm{Flag}(n-k)$. Similarly, the set of sequences $(x_0, x_1, \dots, x_k)$ is isomorphic to $\mathrm{Flag}(k)$.

Denote by ϕ_n the invariant measure on the set $\mathrm{Flag}(n)$, which is computed as follows: The measure τ_n on $G(n,1)$ induces a measure $\widetilde{\tau}_n$ on the set $G(n,n-1)$ via orthogonal duality. If $f(x_0, x_1, \dots, x_n)$ is a simple function on the set $\mathrm{Flag}(n)$, then define

$$\int f\, d\phi_n = \iint f(x_0, x_1, \dots x_n)\, d\phi_{n-1}\,(x_0, \dots, x_{n-1})\, d\widetilde{\tau}_n(x_n)\,, \tag{1.2}$$

so that the measure ϕ_n is inductively defined. It is clearly invariant.

A more explicit form for ϕ_n can be obtained by the following argument: If $(x_0, x_1, \dots, x_n)$ is a flag in $\mathrm{Mod}(n)$, let

$$y_1 = x_1\,, \quad y_2 = x_1^{\perp} \cap x_2\,, \quad y_3 = x_2^{\perp} \cap x_3, \quad \dots, \quad y_n = x_{n-1}^{\perp} \cap x_n\,.$$

The sequence $(y_1, y_2, \dots, y_n)$ is a sequence of orthogonal straight lines, a frame. Conversely, given a frame $(y_1, y_2, \dots, y_n)$, we obtain a flag by setting

$$x_0 = \{0\}\,, \quad x_1 = y_1\,, \quad x_2 = y_1 \vee y_2\,, \quad x_3 = y_1 \vee y_2 \vee y_3\,, \quad \dots\,.$$

This gives a one-to-one correspondence between flags and frames. If

$$f(x_0, x_1, \ldots, x_n)$$

is a real-valued (measurable) function on flags, let $\bar{f}(y_1, y_2, \ldots, y_n)$ be the corresponding function on frames. Then

$$\int f\, d\phi_n = \int\int \cdots \int \bar{f}(y_1, y_2, \ldots, y_n)\, d\tau_1(y_n)\, d\tau_2(y_{n-1}) \cdots d\tau_n(y_1)\,.$$

This result can be read in the simpler language of combinatorics. Having chosen the line $x_1 = y_1$, which can be done in τ_n ways, the subspace x_2 is determined by the choice of a straight line y_2 in the space orthogonal to x_1 (through the origin), which can be done in τ_{n-1} ways, and so on.

The measure of $\mathrm{Flag}(n)$ turns out to be

$$\phi_n(\mathrm{Flag}(n)) = [n][n-1] \cdots [2][1]\,,$$

which is also written $[n]!$, where $[1] = 1$.

We now define an invariant measure on $G(n,k)$. For $A \subseteq G(n,k)$, let $\mathrm{Flag}(A)$ be the set of all flags $(x_0, x_1, \ldots, x_n)$ such that $x_k \in A$. Set

$$\nu_k^n(A) = \frac{1}{[k]![n-k]!}\phi_n(\mathrm{Flag}(A)). \tag{1.3}$$

The measure ν_k^n is evidently invariant under rotation, and we have

$$\nu_k^n(G(n,k)) = \frac{[n]!}{[k]!\,[n-k]!} = \begin{bmatrix} n \\ k \end{bmatrix}. \tag{1.4}$$

These values, called *flag coefficients*, are continuous analogues of the binomial coefficients of the discrete lattice of subsets of a finite set.

Define a measure ν_n on $\mathrm{Mod}(n)$ by taking the direct sum of the measures ν_k^n. That is, for any measurable subset $A \subseteq \mathrm{Mod}(n)$, define

$$\nu_n(A) = \sum_{k=0}^{n} \nu_k^n(A \cap G(n,k))\,.$$

Henceforth, we make the assumption that subsets of $\mathrm{Mod}(n)$ under discussion are measurable with respect to ν_n.

The measure ν_n satisfies the following analogue of the classical Lubell-Yamamoto-Meshalkin (LYM) inequality:

THEOREM 1.1 (The Continuous LYM Inequality) *Let* $A \subseteq \text{Mod}(n)$ *be an antichain. For* $0 \leq k \leq n$ *let*

$$A_k = A \cap G(n, k)\,,$$

so that

$$A = \bigcup_k A_k$$

is a disjoint union. Then

$$\sum_k \frac{\nu_k^n(A_k)}{\begin{bmatrix} n \\ k \end{bmatrix}} \leq 1\,. \tag{1.5}$$

PROOF: For each $0 \leq k \leq n$, the measure of flags meeting A_k is given by

$$\phi_n(\text{Flag}(A_k)) = \nu_k^n(A_k)\,[k]!\,[n-k]!$$

by the definition (1.3) of ν_k^n. Since every flag in $\text{Flag}(n)$ meets A in at most one point, we have

$$\sum_k \nu_k^n(A_k)\,[k]!\,[n-k]! = \sum_k \phi_n(\text{Flag}(A_k)) = \phi_n(\text{Flag}(A)) \leq [n]!$$

It follows that

$$\sum_k \frac{\nu_k^n(A_k)}{\begin{bmatrix} n \\ k \end{bmatrix}} \leq 1\,.$$

∎

The preceding results hold independently of the normalization chosen for the measure τ_n, that is, the value of $[n]$. For example, in [4], Fisk developed a similar construction, in which the total measure of $G(n, 1)$ is taken to be $n\omega_n/2$, the measure suggested by the usual two-to-one quotient map from the unit sphere in $\mathbb{R}^n$. Unfortunately, this normalization does not permit an extension of the lattice analogy, nor does it agree with a related and fundamental normalization for the rigid motion invariant measures on $\mathbb{R}^n$ and its subspaces. To see this and to determine the appropriate choice of normalization for τ_n, we turn briefly to the theory of convex bodies and intrinsic volumes.

2 The Intrinsic Volumes

Denote by $\mathcal{K}^n$ the collection of all compact convex subsets of $\mathbb{R}^n$, that is, n-dimensional Euclidean space. The elements of $\mathcal{K}^n$ are also known as *convex bodies*.

A function $\mu : \mathcal{K}^n \longrightarrow \mathbb{R}$ is called a *valuation* on $\mathcal{K}^n$ if $\mu(\varnothing) = 0$, where $\varnothing$ is the empty set and

$$\mu(K \cup L) = \mu(K) + \mu(L) - \mu(K \cap L)$$

for all $K, L \in \mathcal{K}^n$ such that $K \cup L \in \mathcal{K}^n$ as well.

A valuation μ on $\mathcal{K}^n$ is said to be *continuous* if

$$\lim_{i\to\infty} \mu(K_i) = \mu(K)$$

for any convergent sequence $K_i \longrightarrow K$ in the Hausdorff topology on $\mathcal{K}^n$ (see [17, p. 47]).

Let E_n denote the group of Euclidean (or rigid) motions, that is, the group generated by all translations and rotations in $\mathbb{R}^n$. A valuation μ on $\mathcal{K}^n$ is *invariant under rigid motions*, or simply *invariant*, if $\mu(\phi K) = \mu(K)$ for all $\phi \in E_n$.

A well-known example of a continuous rigid motion invariant valuation on $\mathcal{K}^n$ is the *volume*, denoted V_n. Another example is *surface area*, denoted S. Notice that if P is a rectilinear box with edge lengths $a_1, \ldots, a_n$, then

$$V_n(P) = a_1 a_2 \cdots a_n \,,$$

and

$$S(P) = 2\sum_{i=1}^{n} a_1 \cdots \hat{a}_i \cdots a_n \,,$$

where $\hat{a}_i$ denotes the deletion of a_i from the indicated product. Setting $V_{n-1} = \frac{1}{2}S$, it follows that V_n and V_{n-1} evaluate on a rectilinear box P as the n^{th} and $(n-1)^{\text{th}}$ elementary symmetric polynomials of the edge lengths $a_1, \ldots, a_n$ of P. In fact, it can easily be seen that each of the elementary symmetric polynomials $e_1, \ldots, e_n$ in n variables defines a continuous valuation on the collection of *rectilinear boxes* (with respect to a given ordered orthogonal frame) by evaluating on the edge lengths of such boxes. Moreover, these valuations are evidently invariant under permutations of the edges (or basis elements of the given frame). Therefore, define

$$V_i(P) = e_i(a_1, \ldots, a_n)$$

for all rectilinear boxes P and all $1 \leq i \leq n$. Define $V_0(P)$ to equal 1 if P is nonempty and 0 if P is empty. It turns out that, as is true for the volume V_n and for V_{n-1}, each of the valuations V_i extends to a continuous invariant valuation on all of $\mathcal{K}^n$. In particular, V_0 extends to the Euler characteristic. These (extended) valuations are known as McMullen's *intrinsic volumes* [2, 12, 13][17, p. 210] and have numerous expressions in terms of integral formulae, some of which are described in the following sections.

The intrinsic volume V_i is equal to the i-dimensional volume when restricted to any i-dimensional subspace of $\mathbb{R}^n$. In fact, the intrinsic volumes are normalizations of special cases of *Minkowski mixed volumes,* also known as the *quermassintegrals*. Consequently, the intrinsic volumes satisfy Steiner's formula [17, p. 210] for the volume of Minkowski sums.

Let B_n denote the unit ball in $\mathbb{R}^n$. It follows easily from Steiner's formula that

$$V_i(B_n) = \binom{n}{i} \frac{\omega_n}{\omega_{n-i}} \tag{2.1}$$

where ω_k denotes the k-dimensional volume of the unit ball in $\mathbb{R}^k$.

Hadwiger's characterization theorem states that the intrinsic volumes span the vector space of *all* continuous valuations on $\mathcal{K}^n$ that are invariant under rigid motions (see [2, 5, 8, 13][17, p. 210]).

A valuation μ on $\mathcal{K}^n$ is said to be *homogeneous of degree i* if

$$\mu(\alpha K) = \alpha^i \mu(K)$$

for all $\alpha \geq 0$ and all $K \in \mathcal{K}^n$. It is well-known that, for $0 \leq i \leq n$, the valuation V_i is homogeneous of degree i. Hadwiger's characterization theorem implies that, up to a constant factor, V_i is the *unique* continuous invariant valuation that is invariant under rigid motions and is homogeneous of degree i. In particular, the intrinsic volume V_i is the only continuous and invariant valuation extending i-dimensional volume on $\mathbb{R}^i$ to convex bodies in $\mathbb{R}^n$ for $n > i$.

The intrinsic volumes are related to the invariant measures ν_k^n on the Grassmannians by the *mean projection formulae* [16, p. 221][17, p. 295]). If $K \in \mathcal{K}^n$ and if ξ is a k-dimensional subspace of $\mathbb{R}^n$, denote by $K \mid \xi$ the orthogonal projection of K onto ξ. The mean projection formula (for dimension $k \in \{0, 1, \ldots, n\}$) states that

$$V_k(K) = c_k \int_{G(n,k)} V_k(K \mid \xi)\, d\nu_k^n(\xi)\,, \tag{2.2}$$

for all $K \in \mathcal{K}^n$, where the value of the constant c_k depends on the normalization chosen for the measure ν_k^n. The mean projection formula (2.2) forms a crucial link between measures on the lattice $\mathrm{Mod}(n)$ and the intrinsic volumes on compact convex sets.

For the special case of $k = n - 1$, the formula (2.2) can be expressed in the following form, known as Cauchy's formula:

$$S(K) = \frac{1}{\omega_{n-1}} \int_{\mathbf{S}^{n-1}} V_{n-1}(K \mid u^{\perp})\, du\,.$$

Note that $V_{n-1}(K \mid u^{\perp})$ is just the $(n-1)$-dimensional volume of the projection of K onto the subspace $u^{\perp}$. Since $S(K) = 2V_{n-1}(K)$, we have

$$V_{n-1}(K) = \frac{1}{2\omega_{n-1}} \int_{\mathbf{S}^{n-1}} V_{n-1}(K \mid u^{\perp})\, du\,. \tag{2.3}$$

Alternatively, this result can be expressed as an integral over the projective space $G(n,1)$ (i.e., the set of all lines ℓ through the origin in $\mathbb{R}^n$) rather than an integral over the sphere. Integrating with respect to the Haar *probability* measure on $G(n,1)$, we have

$$V_{n-1}(K) = \alpha \int_{G(n,1)} V_{n-1}(K \mid \ell^{\perp})\, d\ell$$

where α is a constant independent of K. To compute α, set $K = B_n$ to obtain

$$\frac{n\omega_n}{2} = V_{n-1}(B_n) = \alpha \int_{G(n,1)} V_{n-1}(B_n \mid \ell^{\perp})\, d\ell = \alpha\omega_{n-1}$$

so that $\alpha = \frac{n\omega_n}{2\omega_{n-1}}$. We come now to the normalization of the Haar measure on $G(n,1)$, which is the central point of the present work. Denote by τ_n the Haar measure on $G(n,1)$ normalized so that

$$[n] = \tau_n(G(n,1)) = \frac{n\omega_n}{2\omega_{n-1}}\,, \tag{2.4}$$

where ω_n denotes the volume of the unit ball B_n in $\mathbb{R}^n$:

$$\omega_n = \frac{\pi^{n/2}}{\Gamma((n/2)+1)}\,. \tag{2.5}$$

The choice of normalization just described is crucial. For example, (2.3) now becomes

$$V_{n-1}(K) = \int_{G(n,1)} V_{n-1}(K \mid \ell^{\perp})\, d\tau_n\,. \tag{2.6}$$

This is the first of several confirmations that our choice of normalization is the "right" one.

Having chosen a normalization for the measure τ_n on $G(n,1)$, the normalizations for the measures ϕ_n and ν_k^n on $\mathrm{Flag}(n)$ and $G(n,k)$ are then precisely determined by (1.2) and (1.3). Specifically, we have

$$[n]! = \frac{n!\,\omega_n\omega_{n-1}\cdots\omega_1}{2^n\omega_{n-1}\omega_{n-2}\cdots\omega_0} = \frac{n!\,\omega_n}{2^n}\,. \tag{2.7}$$

The flag coefficients now assume a particularly pleasing form. From (1.4) and (2.7) we obtain

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{n!}{k!(n-k)!}\,\frac{\omega_n}{\omega_k\omega_{n-k}} = \binom{n}{k}\frac{\omega_n}{\omega_k\omega_{n-k}}. \tag{2.8}$$

Once again our computations of the flag coefficients differ from the calculations of Fisk [4] due to our special choice of normalization (2.4) for the measure τ_n on $G(n,1)$.

With our choice of normalization (2.4), we have set the factor c_{n-1} from (2.2) equal to 1. This single assignment then determines all other normalizations (2.8) for the measures ν_k^n on $G(n,k)$, where $k \in \{1,\,\ldots,n\}$.

Moreover, the normalizations (2.8) are compatible not only with the intrinsic volume V_{n-1} (see formula (2.6)) but with *all* of the intrinsic volumes. To see this, we compute the rest of the constants c_k in (2.2) by evaluating at $K = B_n$:

$$\begin{aligned} V_k(B_n) = c_k\int_{G(n,k)} V_k(B_n \mid \xi)\,d\nu_k^n(\xi) &= c_k V_k(B_k)\int_{G(n,k)} d\nu_k^n(\xi) \\ &= c_k\omega_k\begin{bmatrix} n \\ k \end{bmatrix}\,. \end{aligned}$$

It now follows from (2.1) and (2.8) that

$$c_k = \frac{V_k(B_n)}{\omega_k}\begin{bmatrix} n \\ k \end{bmatrix}^{-1} = \binom{n}{k}\frac{\omega_n}{\omega_{n-k}\omega_k}\begin{bmatrix} n \\ k \end{bmatrix}^{-1} = 1\,,$$

so that in fact, the formula (2.2) becomes

$$V_k(K) = \int_{G(n,k)} V_k(K \mid \xi)\,d\nu_k^n(\xi)\,, \tag{2.9}$$

for all $k \in \{1,\,\ldots,n\}$.

Recall that V_k is the *unique* continuous invariant valuation defined simultaneously on $\mathcal{K}^n$, for all $n > 0$, so that V_k restricts to the k-dimensional volume on all k-dimensional subspaces of $\mathbb{R}^n$. Recall also that the intrinsic volumes V_k are normalized independently of n; that is, $V_k(L)$ of an l-dimensional convex body L is the same regardless of the dimension $n \geq l$ of the ambient space $\mathbb{R}^n$. The absence of an additional normalizing factor in any of the mean projection formulae (2.9) suggests that our choice of normalization for the measures ν_k^n is indeed the correct one.

When expressed in the language of this section, Kubota's generalization of the mean projection formula [9][17, p. 295] takes the following combinatorially suggestive form:

$$\int_{G(n,l)} V_k(K \mid \xi)\, d\nu_l^n(\xi) = \begin{bmatrix} n-k \\ l-k \end{bmatrix} V_k(K)$$

for $0 \leq k \leq l \leq n$ and all $K \in \mathcal{K}^n$.

The flag coefficients also appear in the principal kinematic formula for convex bodies (see [9], also [16, p. 262] and [17, p. 253]) when expressed in terms of the intrinsic volumes. Let E_n denote the group of Euclidean motions in $\mathbb{R}^n$, the semidirect product of the orthogonal group $O(n)$ (equipped with the Haar probability measure) with the group of translations. The principal kinematic formula then takes the form

$$\int_{E_n} V_0(K \cap gL)\, dg = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}^{-1} V_i(K) V_{n-i}(L)$$

for all $K, L \in \mathcal{K}^n$. Here V_0 denotes zeroth intrinsic volume, usually called the Euler characteristic, which takes the value 0 on the null set and the value 1 on all nonempty, compact, convex sets.

More generally, we have

$$\int_{E_n} V_k(A \cap gK) dg = \sum_{i=0}^{n-k} \begin{bmatrix} i+k \\ k \end{bmatrix} \begin{bmatrix} n \\ i \end{bmatrix}^{-1} V_{k+i}(A) V_{n-i}(K)$$

for $0 \leq k \leq n$ and all $K, L \in \mathcal{K}^n$. Similarly clean formulations also exist for Crofton's formulae (see [9], also [16] and [17, p. 235]).

3 A Sperner Theorem for Subspaces

From now on, all flag coefficients will be normalized as specified in the previous section. Thanks to our choice of normalization, the flag coefficients satisfy a number of properties analogous to those of the binomial coefficients. For example,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ n-k \end{bmatrix}.$$

In analogy to Pascal's triangle relations, we have the following:

PROPOSITION 3.1 *For* $1 \leq k \leq n-1$,

$$\frac{\omega_{k-1}\omega_{n-k}}{\omega_{n-1}} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + \frac{\omega_k \omega_{n-k-1}}{\omega_{n-1}} \begin{bmatrix} n-1 \\ k \end{bmatrix} = \frac{\omega_k \omega_{n-k}}{\omega_n} \begin{bmatrix} n \\ k \end{bmatrix}.$$

PROOF: By direct computation, we have

$$\begin{aligned} &\frac{\omega_{k-1}\omega_{n-k}}{\omega_{n-1}} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + \frac{\omega_k \omega_{n-k-1}}{\omega_{n-1}} \begin{bmatrix} n-1 \\ k \end{bmatrix} \\ &\qquad = \binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k} = \frac{\omega_k \omega_{n-k}}{\omega_n} \begin{bmatrix} n \\ k \end{bmatrix}. \end{aligned}$$

∎

In constructing the flag coefficients, the real numbers $[n]$ play a role analogous to that of the positive integers n in the discrete case of the classical binomial coefficients.

The following proposition demonstrates once again the expedience of our choice of normalization for the measure τ_n.

PROPOSITION 3.2 *The map* $n \longmapsto [n]$ *is an increasing function.*

PROOF: From (2.5) and (1.1) we have

$$[n] = \frac{n\omega_n}{2\omega_{n-1}} = \frac{n\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} = \frac{n\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\frac{n}{2}\Gamma\left(\frac{n}{2}\right)} = \sqrt{\pi} \frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}.$$

Define a function f on the positive real numbers by

$$f(t) = \sqrt{\pi} \frac{\Gamma\left(t+\frac{1}{2}\right)}{\Gamma(t)}.$$

Since $f(\frac{n}{2}) = [n]$, it is sufficient to show that f is an increasing function of t. Recall that [1, p. 15]

$$\Gamma(t) = \lim_{k\to\infty} \frac{k^t k!}{t(t+1)\cdots(t+k)}.$$

This implies that

$$
\begin{aligned}
f(t) &= \sqrt{\pi}\left(\lim_{k\to\infty}\frac{k^{t+\frac{1}{2}}k!}{(t+\frac{1}{2})(t+\frac{1}{2}+1)\cdots(t+\frac{1}{2}+k)}\right) \\
&\quad\times\left(\lim_{k\to\infty}\frac{k^t k!}{t(t+1)\cdots(t+k)}\right)^{-1} \\
&= \sqrt{\pi}\lim_{k\to\infty}\frac{t(t+1)\cdots(t+k)\sqrt{k}}{(t+\frac{1}{2})(t+\frac{1}{2}+1)\cdots(t+\frac{1}{2}+k)}
\end{aligned}
\tag{3.1}
$$

Since the function

$$\frac{t}{t+\frac{1}{2}}$$

is increasing with respect to $t > 0$, so is the product on the right-hand side of (3.1). It follows that f is increasing for $t > 0$, and we conclude that $[n]$ is an increasing function of the positive integers. ∎

The generalized factorial $[n]!$ also satisfies the following property:

PROPOSITION 3.3 *For* $0 \leq k \leq l \leq \frac{n}{2}$,

$$[k]!\,[n-k]! \geq [l]!\,[n-l]!$$

PROOF: To begin, note that

$$0 \leq k \leq l \leq \frac{n}{2} \leq n-l \leq n-k \leq n\,.$$

It follows from Proposition 3.2 that

$$[n-k]\cdots[n-l+1] \geq [l]\cdots[k+1]\,.$$

(Note that there are $l-k$ factors on each side of this identity.) Multiplying on both sides, we obtain

$$[n-k]!\,[k]! \geq [n-l]!\,[l]!$$

∎

The flag coefficients in turn satisfy the following property, in evident analogy to the classical binomial coefficients:

PROPOSITION 3.4 *For* $0 \leq k \leq n$*,*

$$\begin{bmatrix} n \\ k \end{bmatrix} \leq \begin{bmatrix} n \\ \langle \frac{n}{2} \rangle \end{bmatrix} .$$

Here $\langle \frac{n}{2} \rangle$ *denotes the largest integer less than or equal to* $\frac{n}{2}$.

PROOF: Since

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]!\,[n-k]!} = \begin{bmatrix} n \\ n-k \end{bmatrix} ,$$

it is sufficient to consider the case where $k \leq \langle \frac{n}{2} \rangle$. The result then follows immediately from Proposition 3.3 after setting $l = \langle \frac{n}{2} \rangle$. ∎

Our choice of normalization for the measures ν_k^n now enables us to prove a continuous analogue of Sperner's theorem on lattice antichains [6, 11, 18][10, p. 542].

THEOREM 3.5 (Continuous Sperner Theorem) *Assume that* A *is an antichain in* $\mathrm{Mod}(n)$. *Then*

$$\nu_n(A) \leq \begin{bmatrix} n \\ \langle \frac{n}{2} \rangle \end{bmatrix} .$$

PROOF: For $0 \leq k \leq n$, let $A_k = A \cap G(n,k)$. Combining (1.5) with Proposition 3.4, we obtain

$$\sum_k \frac{\nu_k^n(A_k)}{\begin{bmatrix} n \\ \langle \frac{n}{2} \rangle \end{bmatrix}} \leq \sum_k \frac{\nu_k^n(A_k)}{\begin{bmatrix} n \\ k \end{bmatrix}} \leq 1$$

so that

$$\nu_n(A) = \sum_k \nu_k^n(A_k) \leq \begin{bmatrix} n \\ \langle \frac{n}{2} \rangle \end{bmatrix} .$$

∎

A subset $F \subseteq \mathrm{Mod}(n)$ is called an r*-family* if chains in F contain no more than r elements. For example, an antichain is a 1-family. Given an r-family F in $\mathrm{Mod}(n)$, let $F_k = F \cap G(n,k)$. Since every flag in $\mathrm{Mod}(n)$ meets F in at most r elements, we have

$$\sum_{k=0}^{n} \phi_n(\mathrm{Flag}(F_k)) \leq [n]! \cdot r .$$

From (1.3) we then obtain the following generalization of (1.5):

$$\sum_k \frac{\nu_k^n(F_k)}{\left[{n \atop k}\right]} \leq r\,. \tag{3.2}$$

This inequality leads in turn to a continuous analogue of Sperner's theorem for r-families. (For the classical result, see [3, 6] and [10, p. 543].)

THEOREM 3.6 *Let F be an r-family in $\mathrm{Mod}(n)$. Then*

$$\nu_n(F) \leq \left[{n \atop \langle\frac{n+1}{2}\rangle}\right] + \left[{n \atop \langle\frac{n+2}{2}\rangle}\right] + \cdots + \left[{n \atop \langle\frac{n+r}{2}\rangle}\right].$$

Once again $\langle x\rangle$ denotes the largest integer less than or equal to a real number x. In order to prove Theorem 3.6 we make use of the following lemma from [6] (see also [9]):

LEMMA 3.7 *Suppose that $c_0 \geq c_1 \geq \cdots \geq c_n > 0$. If $c_i \geq x_i \geq 0$ for $0 \leq i \leq n$, and if*

$$x_0 + x_1 + \cdots + x_n \geq c_0 + c_1 + \cdots + c_{r-1}\,,$$

then

$$\sum_{k=0}^{n} \frac{x_k}{c_k} \geq r\,.$$

If $c_0 > \cdots > c_n > 0$, then equality holds if and only if $x_i = c_i$ for $0 \leq i \leq r-1$ and $x_i = 0$ for $r \leq i \leq n$.

PROOF OF THEOREM 3.6 Relabel the flag coefficients $c_0, c_1, \ldots, c_n$ in *descending* order, so that $c_0 \geq c_1 \geq \cdots \geq c_n$; then relabel the numerators $\nu_k^n(F_k)$ in (3.2) by $x_0, x_1, \ldots, x_n$, so that each x_k is the numerator of that term of (3.2) having c_k as denominator. The inequality (3.2) now becomes

$$\sum_{k=0}^{n} \frac{x_k}{c_k} \leq r\,.$$

It then follows from Lemma 3.7 that

$$x_0 + x_1 + \cdots + x_n \leq c_0 + c_1 + \cdots + c_{r-1}\,.$$

In other words,

$$\nu_n(F) = \sum_{k=0}^{n} \nu_k^n(F_k) \leq \left[{n \atop \langle\frac{n+1}{2}\rangle}\right] + \left[{n \atop \langle\frac{n+2}{2}\rangle}\right] + \cdots + \left[{n \atop \langle\frac{n+r}{2}\rangle}\right].$$

∎

4 A Multinomial Generalization

In [14] (see also [7]) Meshalkin generalizes the notion of antichain in the finite lattice $P(S)$, defining an *s-system* σ to be a collection of ordered partitions (called *decompositions*) of a finite set S into r blocks, such that the set σ_j, whose elements are the j^{th} blocks of the ordered partitions in σ, is an antichain for each $1 \leq j \leq r$. Meshalkin has shown that the maximum size of an s-system σ of r-decompositions (for fixed r) is given by the maximal multinomial coefficient in r parameters, namely,

$$\binom{n}{\underbrace{\langle n/r\rangle\,,\,\ldots,\langle n/r\rangle}_{r-b},\underbrace{\langle n/r\rangle+1,\,\ldots,\langle n/r\rangle+1}_{b}}$$

where $n \cong r \mod b$ and $0 \leq b \leq r-1$. We now define analogous notions for the lattice $\mathrm{Mod}(n)$.

A map $\delta : \{1, \ldots, r\} \longrightarrow \mathrm{Mod}(n)$ is called an *r-decomposition of* $\mathbb{R}^n$ if

1. $\delta(i) \perp \delta(j)$ for $i \neq j$, and
2. $\delta(1) \oplus \cdots \oplus \delta(r) = \mathbb{R}^n$.

Denote by $\mathrm{Dec}(n, r)$ the set of all r-decompositions of $\mathbb{R}^n$. Note that for each $\delta \in \mathrm{Dec}(n, r)$

$$\dim \delta(1) + \cdots + \dim \delta(r) = n\,.$$

Given positive integers $a_1, a_2, \ldots, a_r$ such that $a_1 + \cdots + a_r = n$ we denote by $\mathrm{Mult}(n; a_1, \ldots, a_r)$ the set of all r-decompositions δ such that $\dim \delta(i) = a_i$ for $i = 1, \ldots, r$. In other words, $\mathrm{Mult}(n; a_1, \ldots, a_r)$ is the set of all (ordered) decompositions of $\mathbb{R}^n$ into direct sums of subspaces having dimensions $a_1, \ldots, a_r$. Evidently the set $\mathrm{Dec}(n, r)$ can be expressed as the finite disjoint union

$$\mathrm{Dec}(n, r) = \biguplus_{a_1+\cdots+a_r=n} \mathrm{Mult}(n; a_1, \ldots, a_r)\,.$$

An s-system of order r is a subset $\sigma \subseteq \mathrm{Dec}(n, r)$ such that the set

$$\{\delta(i) \mid \delta \in \sigma\} \tag{4.1}$$

is an antichain in $\mathrm{Mod}(n)$ for each $1 \leq i \leq r$.

An obvious example of an s-system of order r is $\mathrm{Mult}(n; a_1, \ldots, a_r)$ for some admissible selection of $a_1, \ldots, a_r$. If $\delta, \zeta \in \mathrm{Mult}(n; a_1, \ldots, a_r)$, then

$\delta(i)$ and $\zeta(i)$ both have dimension a_i, so that either $\delta(i) = \zeta(i)$ or the two subspaces are incomparable in the subset partial ordering on $\mathrm{Mod}(n)$. This holds for $i = 1, \ldots, r$, and so the antichain condition on (4.1) is satisfied.

Other examples with which we have already worked are the s-systems of order 2. Let A be an antichain in $\mathrm{Mod}(n)$. For each $V \in A$, express $\mathbb{R}^n$ as the direct sum $V \oplus V^\perp$, so that the pair $(V, V^\perp)$ is a 2-decomposition in $\mathrm{Dec}(n, 2)$. Define

$$A^\perp = \{V^\perp \mid V \in A\} .$$

Then $A^\perp$ is also an antichain in $\mathrm{Mod}(n)$, and the set

$$\sigma = \{(V, V^\perp) \mid V \in A\}$$

is an s-system of order 2. Thus an s-system is in fact a generalization of an antichain. Similarly, the Grassmannian $G(n, k)$ can also be viewed as $\mathrm{Mult}(n; k, n-k)$ through the bijection $V \mapsto (V, V^\perp)$.

In analogy to the construction of the measure ν_k^n on $G(n, k)$, define invariant measures on the sets $\mathrm{Mult}(n; a_1, \ldots, a_r)$ as follows:

For $\delta \in \mathrm{Mult}(n; a_1, \ldots, a_r)$ define a flag $(x_0, x_1, \ldots, x_n) \in \mathrm{Flag}(n)$ to be *compatible* with δ if

1. $x_{a_1} = \delta(1)$ and
2. $x_{a_1+\cdots+a_i}/x_{a_1+\cdots+a_{i-1}} = \delta(i)$ for $i \geq 2$.

Here the quotient $x_{a_1+\cdots+a_i}/x_{a_1+\cdots+a_{i-1}}$ denotes the orthogonal complement of the vector space $x_{a_1+\cdots+a_{i-1}}$ inside the larger space $x_{a_1+\cdots+a_i}$.

For $A \subseteq \mathrm{Mult}(n; a_1, \ldots, a_r)$, let the set of all flags $(x_0, x_1, \ldots, x_n)$ compatible with some $\delta \in A$ be designated by $\mathrm{Flag}(A)$. Define

$$\nu^n_{a_1, a_2, \ldots, a_r}(A) = \frac{1}{[a_1]!\,[a_2]! \cdots [a_r]!}\phi_n(\mathrm{Flag}(A)) . \tag{4.2}$$

To justify this normalization combinatorially, note that to choose a flag compatible with $\delta \in A$ one must choose a frame for each of the vector spaces $\delta(i)$, of which there are $[a_i]!$ choices for each i.

The measure $\nu^n_{a_1, \ldots, a_r}$ is evidently invariant under rotations, and we have

$$\nu^n_{a_1, \ldots, a_r}(\mathrm{Mult}(n; a_1, \ldots, a_r)) = \frac{[n]!}{[a_1]! \cdots [a_r]!} = \begin{bmatrix} n \\ a_1, \ldots, a_r \end{bmatrix} .$$

These values, called *multiflag coefficients*, are continuous analogues of the classical multinomial coefficients.

Define a measure $\nu_{n;r}$ on $\mathrm{Dec}(n,r)$ by taking the direct sum of the measures $\nu^n_{a_1,\ldots,a_r}$. That is, for any measurable subset $A \subseteq \mathrm{Dec}(n,r)$, define

$$\nu_{n;r}(A) = \sum_{a_1+\cdots+a_r=n} \nu^n_{a_1,\ldots,a_r}(A \cap \mathrm{Mult}(n; a_1, \ldots, a_r)) .$$

Just as the continuous Sperner theorem, Theorem 3.5, gives the maximum possible measure for an antichain A in $\mathrm{Mod}(n)$, a generalization of this theorem gives the maximum possible measure for an s-system in $\mathrm{Dec}(n,r)$. En route to such a generalization, we prove a multinomial version of the continuous LYM inequality.

THEOREM 4.1 (Continuous Multinomial LYM Inequality) *Let $\sigma \subseteq \mathrm{Dec}(n,r)$ be an s-system. For $a_1 + \cdots + a_r = n$, let*

$$\sigma_{a_1,\ldots,a_r} = \sigma \cap \mathrm{Mult}(n; a_1, \ldots, a_r)$$

so that

$$\sigma = \bigcup_{a_1+\cdots+a_r=n} \sigma_{a_1,\,\ldots,a_r}$$

is a disjoint union. Then

$$\sum_{a_1+\cdots+a_r=n} \frac{\nu^n_{a_1,\,\ldots,a_r}(\sigma_{a_1,\,\ldots,a_r})}{\begin{bmatrix} n \\ a_1,\,\ldots,a_r \end{bmatrix}} \leq 1 . \tag{4.3}$$

PROOF: For $a_1 + \cdots + a_r = n$, the measure of flags compatible with $\sigma_{a_1,\,\ldots,a_r}$ is given by

$$\phi_n(\mathrm{Flag}(\sigma_{a_1,\,\ldots,a_r})) = \nu^n_{a_1,\,\ldots,a_r}(\sigma_{a_1,\,\ldots,a_r})[a_1]! \cdots [a_r]!$$

by the definition (4.2) of $\nu^n_{a_1,\,\ldots,a_r}$.

Suppose a flag $(x_0, x_1, \ldots, x_n)$ is compatible with both $\gamma, \delta \in \sigma$. Then $\gamma(1) = x_{a_1}$ and $\delta(1) = x_{b_1}$, where $a_1 = \dim \gamma(1)$ and $b_1 = \dim \delta(1)$. Since $(x_0, x_1, \ldots, x_n)$ is a flag, we have $x_{a_1} \subseteq x_{b_1}$ or vice versa. But σ is an s-system, so that either $\gamma(1) = \delta(1)$ or the two spaces are incomparable. Therefore $\gamma(1) = \delta(1)$ and $a_1 = b_1$. Continuing, we have $\gamma(2) = x_{a_1+a_2}/x_{a_1}$ and $\delta(2) = x_{b_1+b_2}/x_{a_1}$ (since $a_1 = b_1$). A similar argument then implies that $\gamma(2) = \delta(2)$ and $a_2 = b_2$. Continuing in this manner, we conclude that $\gamma(i) = \delta(i)$ for each $1 \leq i \leq r$ so that $\gamma = \delta$. In other words, every flag in

$\mathrm{Flag}(n)$ is compatible with at most one r-decomposition $\delta \in \sigma$. It follows that

$$\begin{aligned}\sum_{a_1+\cdots+a_r=n} \nu^n_{a_1,\ldots,a_r}(\sigma_{a_1,\ldots,a_r})[a_1]!\cdots[a_r]! &= \sum_{a_1+\cdots+a_r=n} \phi_n(\mathrm{Flag}(\sigma_{a_1,\ldots,a_r})) \\ &= \phi_n(\mathrm{Flag}(\sigma)) \\ &\le [n]!\,,\end{aligned}$$

so that

$$\sum_{a_1+\cdots+a_r=n} \frac{\nu^n_{a_1,\ldots,a_r}(\sigma_{a_1,\ldots,a_r})}{\left[{n \atop a_1,\ldots,a_r}\right]} \le 1\,.$$

∎

The multiflag coefficients also satisfy the following property, in analogy to the classical multinomial coefficients:

PROPOSITION 4.2 *Let $r \le n$ be positive integers, and suppose that $n = rq + b$, where q is a natural number and $0 \le b \le r-1$ is the integer remainder. For $a_1 + \cdots + a_r = n$,*

$$\begin{bmatrix} n \\ a_1,\ldots,a_r \end{bmatrix} \le \begin{bmatrix} n \\ \underbrace{\langle n/r\rangle,\ldots,\langle n/r\rangle}_{r-b},\underbrace{\langle n/r\rangle+1,\ldots,\langle n/r\rangle+1}_{b} \end{bmatrix}.$$

PROOF: Let $a_1,\ldots,a_r$ be positive integers such that $a_1+\cdots+a_r=n$. Without loss of generality, suppose that $a_1 < \left\langle \frac{n}{r}\right\rangle$. Then $a_i > \left\langle \frac{n}{r}\right\rangle$ for some $i > 1$. Again without loss of generality, suppose that

$$a_1 < \left\langle \frac{n}{r} \right\rangle < a_2\,.$$

Then $a_2 - a_1 \ge 2$, so that

$$a_1 < \left\langle \frac{a_1+a_2}{2} \right\rangle < a_2\,.$$

It then follows from Proposition 3.3 that

$$[a_1+1]!\,[a_2-1]! \le [a_1]!\,[a_2]!\,.$$

Replace a_1 with a_1+1 and a_2 with a_2-1. Note that the identity $a_1+\cdots+a_r = n$ is preserved. This process is repeated until $a_i \ge \left\langle \frac{n}{r}\right\rangle$ for all $1 \le i \le r$, that is, until $a_i = \left\langle \frac{n}{r} \right\rangle + 1$ for $1 \leq i \leq b$ and $a_i = \left\langle \frac{n}{r} \right\rangle$ for $b + 1 \leq i \leq r$, where b is the integer remainder upon division of n by r.

Since each iteration of this procedure decreases the value of the product $[a_1]! \cdots [a_r]!$, it follows that

$$[a_1]! \cdots [a_r]! \geq ([\langle n/r \rangle]!)^{r-b} ([\langle n/r \rangle + 1]!)^b$$

for all $a_1 + \cdots + a_r = n$. Therefore,

$$\frac{[n]!}{[a_1]! \cdots [a_r]!} \leq \frac{[n]!}{([\langle n/r \rangle]!)^{r-b} ([\langle n/r \rangle + 1]!)^b} ,$$

for all $a_1 + \cdots + a_r = n$. ∎

We are now able to prove a continuous analogue to Meshalkin's theorem [7, 14], a multinomial generalization of the continuous Sperner theorem (Theorem 3.5).

THEOREM 4.3 (Continuous Meshalkin Theorem) *Let σ be an s-system in* $\mathrm{Dec}(n, r)$. *Then*

$$\nu_{n;r}(\sigma) \leq \left[\begin{matrix} n \\ \underbrace{\langle n/r \rangle , \ldots , \langle n/r \rangle}_{r-b}, \underbrace{\langle n/r \rangle + 1, \ldots , \langle n/r \rangle + 1}_{b} \end{matrix} \right],$$

where $n \cong b \mod r$.

PROOF: We reason in analogy to the proof of Meshalkin's theorem due to Hochberg and Hirsch [7]. For $a_1 + \cdots + a_r = n$, let $\sigma_{a_1,\ldots,a_r} = \sigma \cap \mathrm{Mult}(n; a_1, \ldots, a_r)$. Combining (4.3) and Proposition 4.2, we obtain

$$\sum_{a_1+\cdots+a_r=n} \frac{\nu^n_{a_1,\ldots,a_r}(\sigma_{a_1,\ldots,a_r})}{\left[\begin{smallmatrix} n \\ \langle n/r \rangle, \ldots, \langle n/r \rangle, \langle n/r \rangle + 1, \ldots, \langle n/r \rangle + 1 \end{smallmatrix} \right]} \leq \sum_{a_1+\cdots+a_r=n} \frac{\nu^n_{a_1,\ldots,a_r}(\sigma_{a_1,\ldots,a_r})}{\left[\begin{smallmatrix} n \\ a_1, \ldots, a_r \end{smallmatrix} \right]} \leq 1$$

so that

$$\nu_{n;r}(\sigma) = \sum_{a_1+\cdots+a_r=n} \nu^n_{a_1,\ldots,a_r}(\sigma_{a_1,\ldots,a_r}) \leq \left[\begin{matrix} n \\ \langle n/r \rangle , \ldots , \langle n/r \rangle , \langle n/r \rangle + 1, \ldots , \langle n/r \rangle + 1 \end{matrix} \right].$$

∎

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