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Covering shadows with a smaller volume

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Abstract

For $n \geqslant 2$ a construction is given for convex bodies K and L in $\mathbb{R}^n$ such that the orthogonal projection L_u onto the subspace $u^\perp$ contains a translate of K_u for every direction u, while the volumes of K and L satisfy $V_n(K) > V_n(L)$.

A more general construction is then given for n-dimensional convex bodies K and L such that each orthogonal projection L_ξ onto a k-dimensional subspace ξ contains a translate of K_ξ, while the mth intrinsic volumes of K and L satisfy $V_m(K) > V_m(L)$ for all $m > k$.

For each $k = 1, \ldots, n$, we then define the collection $\mathcal{C}_{n,k}$ to be the closure (under the Hausdorff topology) of all Blaschke combinations of suitably defined cylinder sets (prisms).

It is subsequently shown that, if $L \in \mathcal{C}_{n,k}$, and if the orthogonal projection L_ξ contains a translate of K_ξ for every k-dimensional subspace ξ of $\mathbb{R}^n$, then $V_n(K) \leqslant V_n(L)$.

The families $\mathcal{C}_{n,k}$, called k-cylinder bodies of $\mathbb{R}^n$, form a strictly increasing chain

$$\mathcal{C}_{n,1} \subset \mathcal{C}_{n,2} \subset \cdots \subset \mathcal{C}_{n,n-1} \subset \mathcal{C}_{n,n},$$

where $\mathcal{C}_{n,1}$ is precisely the collection of centrally symmetric compact convex sets in $\mathbb{R}^n$, while $\mathcal{C}_{n,n}$ is the collection of all compact convex sets in $\mathbb{R}^n$. Members of each family $\mathcal{C}_{n,k}$ are seen to play a fundamental role in relating covering conditions for projections to the theory of mixed volumes, and members of $\mathcal{C}_{n,k}$ are shown to satisfy certain geometric inequalities. Related open questions are also posed.

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Suppose that K and L are compact convex subsets of n-dimensional Euclidean space. For a given dimension $1 \leqslant k < n$, suppose that every k-dimensional orthogonal projection (shadow) of K can be translated inside the corresponding projection of L. Does it follow that K has smaller volume than L? In this article it is shown that the answer in general is *no*. It is then shown that the answer is *yes* if L is chosen from a suitable family of convex bodies that includes certain cylinders and other sets with a direct sum decomposition.

Many inverse questions from convex and integral geometry take the following form: Given two convex bodies K and L, and two geometric invariants f and g (such as volume, or surface area, or some measure of sections or projections), does $f(K) \leqslant f(L)$ imply $g(K) \leqslant g(L)$? If not, then what additional conditions on K and L are necessary?

These questions are motivated in part by the projection theorems of Groemer [7], Hadwiger [9], and Rogers [17]. In particular, if two compact convex sets have translation congruent (or, more generally, homothetic) projections in every linear subspace of some chosen dimension $k \geqslant 2$, then the original sets K and L must be translation congruent (or homothetic). Rogers also proved analogous results for sections of sets with hyperplanes through a base point [17]. These results then set the stage for more general (and often much more difficult) questions, in which the rigid conditions of translation congruence or homothety are replaced with weaker conditions, such as containment up to translation, inequalities of measure, etc.

Two notorious questions of this kind are the Shephard Problem [20] (solved independently by Petty [16] and Schneider [18]), and the Busemann–Petty Problem [2] (solved in work of Gardner [3,6], Schlumprecht [6], Koldobsky [6,13], and Zhang [22,23]). Both questions address properties of bodies K and L that are assumed to be *centrally symmetric* about the origin.

The Shephard Problem asks: if the $(n-1)$-dimensional volumes of the orthogonal projections K_u and L_u of convex bodies K and L onto the subspace $u^\perp$ satisfy the volume inequality $V_{n-1}(K_u) \leqslant V_{n-1}(L_u)$ for every direction u, does it follow that $V_n(K) \leqslant V_n(L)$? Although there are ready counterexamples for general (possibly non-symmetric) convex bodies, the problem is more difficult to address under the stated assumption that K and L are both centrally symmetric. In this case Petty and Schneider have shown that, while the answer in general is still no for dimensions $n \geqslant 3$, the answer is yes when the convex set L is a projection body; that is, an origin-symmetric zonoid.

The Busemann–Petty Problem addresses the analogous question for sections through the origin. Suppose that convex bodies K and L are centrally symmetric about the origin. If we assume that the $(n-1)$-dimensional sections of K and L satisfy the volume inequality

$$V_{n-1}\bigl(K \cap u^\perp\bigr) \leqslant V_{n-1}\bigl(L \cap u^\perp\bigr)$$

for every direction u, does it follow that $V_n(K) \leqslant V_n(L)$? Surprisingly the answer is *no* for bodies of dimension $n \geqslant 5$ and *yes* for bodies of dimension $n \leqslant 4$ (see [3,6,22,23]). Moreover, Lutwak [14] has shown that, in analogy to the Petty–Schneider theorem, the answer is always *yes* when the set L is an *intersection body*, a construct highly analogous to projection bodies (zonoids), but for which projection (the cosine transform) is replaced in the construction with intersection (the Radon transform, respectively). A more complete discussion of background to the Busemann–Petty Problem, its solution, and its variations (some of which remain open), can be found in the comprehensive book by Gardner [5].

Both of the previous problems assume that the bodies in question are either centrally symmetric or symmetric about the origin; that is, $K = -K$ and $L = -L$ (up to translation). If this elementary assumption is omitted, then both questions are easily seen to have negative answers. For the projection problem, compare the Reuleaux triangle, and its higher-dimensional analogues, with the Euclidean ball, or compare any non-centered convex body with its Blaschke body [5, p. 116]. For the intersection problem, consider a non-centered planar set having an equichordal point, or the dual analogue of the Blaschke body of a non-centered set (see [5, p. 117] or [14]).

In the present article we consider a related, but fundamentally different, family of questions.

Suppose that, instead of comparing the areas of the projections of K and L, we assume that the projections of L *contain translates* of the projections of K. Specifically, suppose that, for each direction u, the orthogonal projection L_u of L contains a translate of the corresponding projection K_u (although the required translation may vary depending on u). Does it follow that L contains a translate of K? Does it even follow that $V_n(K) \leqslant V_n(L)$?

These questions have easily described negative answers in dimension 2, since the projections are 1-dimensional, and convex 1-dimensional sets have very little structure. (Once again, consider the Reuleaux triangle and the circle.) The interesting cases begin when comparing 2-dimensional projections of 3-dimensional objects, and continue from there.

For higher dimensions, a simple example illustrates once again that K might *not* fit inside L, even though every projection of L can be translated to cover the corresponding projection of K. Let L denote the unit Euclidean 3-ball, and let K denote the regular tetrahedron having edge length $\sqrt{3}$. Jung's Theorem [1, p. 84], [21, p. 320] implies that every 2-projection of K is covered by a translate of the unit disk. But a simple computation shows that L does not contain a translate of the tetrahedron K. An analogous construction yields a similar result for higher-dimensional simplices and Euclidean balls. One might say that, although K can "hide behind" L from any observer's perspective, this does not imply that K can hide *inside* L. The question of what *additional* conditions on K and L guarantee the covering of K by a translate of L is addressed in [10,11].

In the previous counterexample, meanwhile, it is still the case that the set L having larger (covering) shadows also has *larger volume* than K. Although the question of comparing volumes is more subtle, there are counterexamples to this property as well.

This article presents the following results for every dimension $n \geqslant 2$:

1. There exist n-dimensional convex bodies K and L such that the orthogonal projection L_u contains a translate of K_u for every direction u, while $V_n(K) > V_n(L)$.
2. There is a large class of bodies $\mathcal{C}_{n,n-1}$ such that, if $L \in \mathcal{C}_{n,n-1}$ and if L_u contains a translate of K_u for every direction u, then $V_n(K) \leqslant V_n(L)$.

In particular, it will be shown that if the body L having covering shadows is a *cylinder*, then $V_n(K) \leqslant V_n(L)$. The more general collection $\mathcal{C}_{n,n-1}$, called $(n-1)$-cylinder bodies, play a role for the covering projection problem in analogy to that of intersection bodies for the Busemann–Petty Problem and that of zonoids for the Shephard Problem.

These results generalize to questions about shadows (projections) of arbitrary lower dimension. If ξ is a k-dimensional subspace of $\mathbb{R}^n$, denote by K_ξ the orthogonal projection of a body K into ξ. For convex bodies K in $\mathbb{R}^n$ and $0 \leqslant m \leqslant n$, denote by $V_m(K)$ the mth *intrinsic volume* of K. The main theorems of this article also yield the following more general observations, for each $n \geqslant 2$ and each $1 \leqslant k \leqslant n-1$:

1′. There exist n-dimensional convex bodies K and L such that the orthogonal projection L_ξ contains a translate of K_ξ for each k-dimensional subspace ξ of $\mathbb{R}^n$, while $V_m(K) > V_m(L)$ for all $m > k$.

2′. There is a class of bodies $\mathcal{C}_{n,k}$ such that, if $L \in \mathcal{C}_{n,k}$ and if the orthogonal projection L_ξ contains a translate of K_ξ for each k-dimensional subspace ξ of $\mathbb{R}^n$, then $V_n(K) \leqslant V_n(L)$.

The aforementioned counterexamples are constructed in Sections 2 and 3. Cylinder bodies and their relation to projection and covering are described in Sections 4 and 5, leading to the Shadow Containment Theorem 5.3, which relates covering of shadows to a family of inequalities for mixed volumes. These developments lead in turn to Theorem 6.1, where it is shown that, if every shadow of a cylinder body L contains a translate of the corresponding shadow of K, then L must have greater volume than K. In Section 7 the counterexample constructions of Sections 2 and 3 are used to prove a family of geometric inequalities satisfied by members of each collection $\mathcal{C}_{n,k}$. Section 8 uses Theorem 6.1 to prove that $V_n(K) \leqslant nV_n(L)$ whenever the projections of K can be translated inside those of L.

The constructions and theorems of this article motivate a number of open questions related to covering projections, some of which are posed in the final section.

1. Preliminary background

Denote by $\mathcal{K}_n$ the set of compact convex subsets of $\mathbb{R}^n$. The n-dimensional (Euclidean) volume of a convex set K will be denoted $V_n(K)$. If u is a unit vector in $\mathbb{R}^n$, denote by K_u the orthogonal projection of a set K onto the subspace $u^\perp$.

Let $h_K : \mathbb{R}^n \to \mathbb{R}$ denote the support function of a compact convex set K; that is,

$$h_K(v) = \max_{x \in K} x \cdot v.$$

If u is a unit vector in $\mathbb{R}^n$, denote by K^u the support set of K in the direction of u; that is,

$$K^u = \big\{x \in K \mid x \cdot u = h_K(u)\big\}.$$

If P is a convex polytope, then P^u is the maximal face of P having u in its outer normal cone.

Given two compact convex sets $K, L \in \mathcal{K}_n$ and $a, b \geqslant 0$ denote

$$aK + bL = \{ax + by \mid x \in K \text{ and } y \in L\}.$$

An expression of this form is called a *Minkowski combination* or *Minkowski sum*. Because K and L are convex, the set $aK + bL$ is also convex. Convexity also implies that $aK + bK = (a+b)K$ for all $a, b \geqslant 0$.

Support functions are easily seen to satisfy the identity $h_{aK+bL} = ah_K + bh_L$. Moreover, the volume of a Minkowski combination of two compact convex sets is given by *Steiner's formula:*

$$V_n(aK + bL) = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i V_{n-i,i}(K, L), \tag{1}$$

where the *mixed volumes* $V_{n-i,i}(K, L)$ depend only on K and L and the indices i and n. In particular, if we fix two convex sets K and L then the function $f(a, b) = V_n(aK + bL)$ is a homogeneous polynomial of degree n in the non-negative variables a, b.

Each mixed volume $V_{n-i,i}(K, L)$ is non-negative, continuous in the entries K and L, and monotonic with respect to set inclusion. Note also that $V_{n-i,i}(K, K) = V_n(K)$. If ψ is an affine transformation whose linear component has determinant denoted $\det \psi$, then $V_{i,n-i}(\psi K, \psi L) = |\det \psi| V_{n-i,i}(K, L)$.

If P is a polytope with non-empty interior in $\mathbb{R}^n$, a *facet* of P is a face (support set) of P having dimension $n-1$. The mixed volume $V_{n-1,1}(P, K)$ satisfies the classical "base-height" formula

$$V_{n-1,1}(P, K) = \frac{1}{n} \sum_{u \perp \partial P} h_K(u) V_{n-1}\big(P^u\big), \tag{2}$$

where this sum is finite, taken over all outer normals u to the facets on the boundary ∂P. These and many other properties of convex bodies and mixed volumes are described in detail in each of [1,19,21].

The *Brunn–Minkowski inequality* asserts that, for $0 \leqslant \lambda \leqslant 1$,

$$V_n\big((1-\lambda)K + \lambda L\big)^{1/n} \geqslant (1-\lambda) V_n(K)^{1/n} + \lambda V_n(L)^{1/n}. \tag{3}$$

If K and L have interior, then equality holds in (3) if and only if K and L are homothetic; that is, iff there exist $a > 0$ and $x \in \mathbb{R}^n$ such that $L = aK + x$. On combining (3) with Steiner's formula (1) one obtains the *Minkowski mixed volume inequality*:

$$V_{n-1,1}(K, L)^n \geqslant V_n(K)^{n-1} V_n(L), \tag{4}$$

with the same equality conditions as in (3). See, for example, any of [1, p. 98], [4], [5, p. 417], [19, p. 317], [21, p. 300].

If $K \in \mathscr{K}_n$ has non-empty interior, define the *surface area measure* S_K on the $(n-1)$-dimensional unit sphere $\mathbb{S}^{n-1}$ as follows. For $A \subseteq \mathbb{S}^{n-1}$ denote by $K^A = \bigcup_{u \in A} K^u$, and define $S_K(A) = \mathcal{H}^{n-1}(K^A)$, the $(n-1)$-dimensional Hausdorff measure of the subset K^A of the boundary of K. (See [19, p. 203].)

If P is a polytope, then S_P is a discrete measure concentrated at precisely those directions u that are outer normals to the facets of P.

The measure S_K is easily shown to satisfy the property that

$$\int_{\mathbb{S}^{n-1}} u \, dS_K = \vec{o}, \tag{5}$$

that is, the mass distribution on the sphere described by S_K has center of mass at the origin. The identity (2) can now be expressed in its more general form:

$$V_{n-1,1}(K, L) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_L(u) \, dS_K(u), \tag{6}$$

for all convex bodies K and L such that K has non-empty interior. It follows from (6) and the Minkowski linearity of the support function that, for $K, L, M \in \mathscr{K}_n$ and $a, b \geqslant 0$,

$$V_{n-1,1}(K, aL + bM) = a V_{n-1,1}(K, L) + b V_{n-1,1}(K, M). \tag{7}$$

Let B_n denote the n-dimensional Euclidean ball centered at the origin and having unit radius. Since $h_{B_n} = 1$ in every direction, it follows that $nV_{n-1,1}(K, B_n) = S(K)$, the surface area of the convex body K.

Minkowski's Existence Theorem [1, p. 125], [5, p. 400], [19, p. 390] gives an important and useful converse to the identity (5): If μ is a non-negative measure on the unit sphere $\mathbb{S}^{n-1}$ such that μ has center of mass at the origin, and if μ is not concentrated on any great (equatorial) $(n-1)$-subsphere, then $\mu = S_K$ for some $K \in \mathscr{K}_n$. Moreover, this convex body K is *unique up to translation*.

Minkowski's Existence Theorem provides the framework for the following definition: For $K, L \in \mathscr{K}_n$ and $a, b \geqslant 0$, define the *Blaschke combination* $a \cdot K \# b \cdot L$ to be the unique convex body (up to translation) such that

$$S_{a\cdot K \# b\cdot L} = aS_K + bS_L.$$

Although the Blaschke sum $K \# L$ is identical (up to translation) to the Minkowski sum $K + L$ for convex bodies K and L in $\mathbb{R}^2$, the two sums are substantially different for bodies in $\mathbb{R}^n$ where $n \geqslant 3$. Moreover, for dimension $n \geqslant 3$, the scalar multiplication $a \cdot K$ also differs from the usual scalar multiplication aK used with Minkowski combinations. Specifically, $a \cdot K = a^{\frac{1}{n-1}} K$, since surface area in $\mathbb{R}^n$ is homogeneous of degree $n - 1$.

It follows from (6) that, for $K, L, M \in \mathscr{K}_n$ and $a, b \geqslant 0$,

$$V_{n-1,1}(a \cdot K \# b \cdot L, M) = aV_{n-1,1}(K, M) + bV_{n-1,1}(L, M). \tag{8}$$

Note the important difference between (7) and (8) for $n \geqslant 3$.

It is not difficult to show that every polytope with interior having generic facet normals is a Blaschke combination of a finite number of simplices, while every centrally symmetric polytope with interior is Blaschke combination of a finite number of parallelotopes (i.e., affine images of cubes) [8, p. 334]. A standard continuity argument (using the Minkowski Existence Theorem and the selection principle for convex bodies [19, p. 50]) then implies that every convex body can be approximated (in the Hausdorff topology) by Blaschke combinations of simplices, while every centrally symmetric convex body can be approximated by Blaschke combinations of parallelotopes.

A brief and elegant discussion of Blaschke sums, their properties, and applications, can also be found in [14].

2. A counterexample

We will exhibit convex bodies K and L in $\mathbb{R}^n$ such that $V_n(K) > V_n(L)$, while the orthogonal projection L_u contains a translate of the corresponding projection K_u for each unit direction u.

Note that a suitable disk and Reuleaux triangle provide a well-known example in the 2-dimensional case. This section provides examples for bodies of dimension $n \geqslant 3$.

For $K \in \mathscr{K}_n$, denote by r_K the *inradius* of K; that is, the maximum radius taken over all Euclidean balls inside K. Denote by d_K the *minimal width* of K; that is, the minimum length taken over all orthogonal projections of K onto lines through the origin. The minimal width is also equal to the minimum distance between any two parallel support planes for K.

Let Δ denote the n-dimensional regular simplex having unit edge length. The following well-known quantities will be used in the construction that follows:

$$\begin{aligned} \tau_n &= \text{volume of } \Delta &&= \frac{\sqrt{n+1}}{2^{n/2} n!}, \\ S(\Delta) &= \text{surface area of } \Delta &&= \frac{(n+1)\sqrt{n}}{2^{\frac{n-1}{2}}(n-1)!} = (n+1)\tau_{n-1}, \\ r_\Delta &= \text{inradius of } \Delta &&= \frac{1}{\sqrt{2n(n+1)}} = \frac{n\tau_n}{S(\Delta)}, \end{aligned}$$

and

$$d_\Delta = \text{ minimal width of } \Delta = \begin{cases} \frac{2(n+1)}{\sqrt{n+2}} r_\Delta, \\ 2\sqrt{n} r_\Delta \end{cases} = \begin{cases} \sqrt{\frac{2(n+1)}{n(n+2)}} & \text{if } n \text{ is even}, \\ \sqrt{\frac{2}{n+1}} & \text{if } n \text{ is odd}. \end{cases} \tag{9}$$

See, for example, [1, p. 86].

To construct and verify the counterexample it will be necessary to compare the minimal width and inradius of a regular simplex with those of its lower-dimensional projections. *Steinhagen's inequality* asserts that, for $K \in \mathscr{K}_n$,

$$r_K \geqslant \begin{cases} \frac{\sqrt{n+2}}{2n+2} d_K & \text{if } n \text{ is even}, \\ \frac{1}{2\sqrt{n}} d_K & \text{if } n \text{ is odd}. \end{cases} \tag{10}$$

A proof of (10) is given in [1, p. 86]. If u is a unit vector, then the orthogonal projection Δ_u satisfies $d_{\Delta_u} \geqslant d_\Delta$, where d_{Δ_u} is now computed from within the $(n-1)$-dimensional subspace $u^\perp$. Since $\dim(\Delta_u) = n-1$ has parity opposite that of n, it follows from (10) that

$$r_{\Delta_u} \geqslant \begin{cases} \frac{\sqrt{n+1}}{2n} d_{\Delta_u} \\ \frac{1}{2\sqrt{n-1}} d_{\Delta_u} \end{cases} \geqslant \begin{cases} \frac{\sqrt{n+1}}{2n} d_\Delta & \text{if } n \text{ is odd}, \\ \frac{1}{2\sqrt{n-1}} d_\Delta & \text{if } n \text{ is even}. \end{cases}$$

Combining this with (9) yields

$$r_{\Delta_u} \geqslant \left\{ \begin{array}{ll} \frac{1}{n\sqrt{2}} & \text{if } n \text{ is odd} \\ \frac{\sqrt{n+1}}{\sqrt{2}\sqrt{n(n-1)(n+2)}} & \text{if } n \text{ is even} \end{array} \right\} \geqslant \frac{1}{n\sqrt{2}}. \tag{11}$$

For $0 \leqslant \epsilon \leqslant 1$, denote

$$K^\epsilon = \epsilon\Delta + \left(\frac{1-\epsilon}{n\sqrt{2}}\right) B_n.$$

Proposition 2.1. *For each unit vector u in $\mathbb{R}^n$, there exists $v \in u^\perp$ such that*

$$K_u^\epsilon + v \subseteq \Delta_u.$$

In other words, each shadow of the simplex Δ contains a translate of the corresponding shadow of K^ϵ.

Proof. Let u be a unit vector in $\mathbb{R}^n$. Since $\frac{1}{n\sqrt{2}} \leqslant r_{\Delta_u}$, there exists $w \in u^\perp$ such that

$$\frac{1}{n\sqrt{2}} B_{n-1} \subseteq \Delta_u - w.$$

Hence,

$$K_u^\epsilon = \epsilon \Delta_u + (1-\epsilon)\frac{1}{n\sqrt{2}} B_{n-1} \subseteq \epsilon \Delta_u + (1-\epsilon)(\Delta_u - w) = \Delta_u + (\epsilon - 1)w.$$

Setting $v = (1-\epsilon)w$, we have $K_u^\epsilon + v \subseteq \Delta_u$. $\square$

Next, recall from Steiner's formula (1) that if K is a convex body in $\mathbb{R}^n$ then

$$V_n(\epsilon K + \alpha B_n) = \epsilon^n V_n(K) + \epsilon^{n-1}\alpha S(K) + \alpha^2 f(\alpha, \epsilon), \tag{12}$$

where $f(\alpha, \epsilon)$ is a polynomial in α and ϵ having non-negative coefficients.

Proposition 2.2. *If $1 - \epsilon > 0$ is sufficiently small, then $V_n(K^\epsilon) > V_n(\Delta)$.*

Proof. We need to show that $V_n(K^\epsilon) - V_n(\Delta) > 0$. Applying (12) yields

$$\begin{aligned}
V_n\big(K^\epsilon\big) - V_n(\Delta) &= V_n\left(\epsilon\Delta + \frac{1-\epsilon}{n\sqrt{2}} B_n\right) - V_n(\Delta) \\
&= \big(\epsilon^n - 1\big)V_n(\Delta) + \epsilon^{n-1}\left(\frac{1-\epsilon}{n\sqrt{2}}\right)S(\Delta) + \left(\frac{1-\epsilon}{n\sqrt{2}}\right)^2 f_n(\epsilon) \\
&= \big(\epsilon^n - 1\big)\left(\frac{\sqrt{n+1}}{2^{n/2}n!}\right) + \epsilon^{n-1}\left(\frac{1-\epsilon}{n\sqrt{2}}\right)\left(\frac{(n+1)\sqrt{n}}{2^{\frac{n-1}{2}}(n-1)!}\right) + \left(\frac{1-\epsilon}{n\sqrt{2}}\right)^2 f_n(\epsilon) \\
&= \big(\epsilon^n - 1\big)\left(\frac{\sqrt{n+1}}{2^{n/2}n!}\right) + \epsilon^{n-1}(1-\epsilon)\left(\frac{(n+1)\sqrt{n}}{2^{n/2}n!}\right) + \left(\frac{1-\epsilon}{n\sqrt{2}}\right)^2 f_n(\epsilon)
\end{aligned}$$

where $f_n(\epsilon)$ is a polynomial in ϵ.

It follows that $V_n(K^\epsilon) - V_n(\Delta) > 0$ if and only if

$$\epsilon^{n-1}(1-\epsilon)\left(\frac{(n+1)\sqrt{n}}{2^{n/2}n!}\right) + \left(\frac{(1-\epsilon)^2}{2n^2}\right) f_n(\epsilon) > \big(1-\epsilon^n\big)\left(\frac{\sqrt{n+1}}{2^{n/2}n!}\right)$$

if and only if

$$\epsilon^{n-1}\sqrt{n(n+1)}+2^{n/2}n!\left(\frac{1-\epsilon}{2n^2\sqrt{n+1}}\right)f_n(\epsilon)>\left(1+\epsilon+\epsilon^2+\cdots+\epsilon^{n-1}\right). \tag{13}$$

As $\epsilon \to 1$, the left-hand side of (13) approaches $\sqrt{n(n+1)}$, while the right-hand side approaches n, a strictly smaller value for all positive integers n. It follows that $V_n(K_\epsilon) > V_n(\Delta)$ for ϵ sufficiently close to 1. □

Propositions 2.1 and 2.2 imply that if $0 < \epsilon < 1$ is sufficiently close to 1, then every shadow of Δ contains a translate of the corresponding shadow of the body K^ϵ, even though $V_n(K^\epsilon) > V_n(\Delta)$.

More precise conditions on admissible values of ϵ depend on n. For the case $n = 3$ the inequalities used in the proof of Proposition 2.2, along with some additional very crude estimates, imply that $\epsilon = 0.9$ gives a specific counterexample. In other words, the 3-dimensional convex bodies

$$K = \frac{9}{10}\Delta_3 + \frac{1}{30\sqrt{2}}B_3 \quad \text{and} \quad \Delta_3$$

have the property that each shadow of the unit regular tetrahedron Δ_3 contains a translate of the corresponding shadow of K, even though K has *greater* volume than Δ_3. (A calculation yields $V_3(K) \approx 0.122$ and $V_3(\Delta_3) \approx 0.118$.)

At this point one might ask whether suitable conditions on either of the bodies K and L might guarantee that covering shadows implies larger volume. It is not difficult to show that if L is centrally symmetric, then L will have greater volume than K when the shadows of L can cover those of K. To see this, suppose that $L = -L$. If $K_u \subseteq L_u + v$, then $-K_u \subseteq -L_u - v = L_u - v$ so that

$$K_u + (-K_u) \subseteq L_u + v + L_u - v = 2L_u$$

for every direction u. It follows that $K + (-K) \subseteq L + L = 2L$. Monotonicity of volume and the Brunn–Minkowski Inequality (3) then imply that

$$V_n(L)^{1/n} \geqslant V_n\left(\frac{1}{2}K + \frac{1}{2}(-K)\right)^{1/n} \geqslant \frac{1}{2}V_n(K)^{1/n} + \frac{1}{2}V_n(-K)^{1/n} = V_n(K)^{1/n},$$

so that $V_n(L) \geqslant V_n(K)$.

This volume inequality also turns out to hold when L is chosen from a much larger family of bodies, to be described in Section 6.

3. A more general counterexample

The counterexample of Section 2 will now be generalized. If ξ is a k-dimensional subspace of $\mathbb{R}^n$, denote by K_ξ the orthogonal projection of a set $K \subseteq \mathbb{R}^n$ to the subspace ξ. For $0 \leqslant m \leqslant n$ denote by $V_m(K)$ the mth intrinsic volume of K. The intrinsic volume functional V_m restricts to m-dimensional volume on m-dimensional convex sets and is proportional to the mean m-volume of the m-dimensional orthogonal projections of K for more general $K \in \mathscr{K}_n$. See, for example, [12, p. 125] or [19, p. 210].

The following lemma is helpful for extending some low-dimensional constructions to higher dimension.

Lemma 3.1. *Suppose that K and L are compact convex sets in $\mathbb{R}^j \subseteq \mathbb{R}^n$, where $j \leqslant n$, and suppose that L_ξ contains a translate of K_ξ for each i-subspace ξ of $\mathbb{R}^j$. Then L_ξ contains a translate of K_ξ for each i-subspace ξ of $\mathbb{R}^n$.*

In other words, if the i-dimensional shadows of L can cover those of K in $\mathbb{R}^j$, then this covering relation is preserved when K and L are embedded together (along with $\mathbb{R}^j$) in the higher-dimensional space $\mathbb{R}^n$.

Proof. Suppose that K and L are compact convex sets in $\mathbb{R}^j$, and suppose that L_ξ contains a translate of K_ξ for each i-subspace ξ of $\mathbb{R}^j$.

Suppose that η is an i-dimensional subspace of $\mathbb{R}^{j+1}$. Then $\dim(\eta^\perp) = j - i + 1$, and $\dim(\eta^\perp \cap \mathbb{R}^j) = j - i$ for generic choices of η. Assume η is chosen this way.

Let ξ denote the orthogonal complement of $\eta^\perp \cap \mathbb{R}^j$ taken within $\mathbb{R}^j$. Since $\dim(\xi) = i$, there exists $v \in \mathbb{R}^j$ such that $(K + v)_\xi \subseteq L_\xi$, by the covering assumption for K and L in $\mathbb{R}^j$. This means that, for each $x \in K + v$, there exists $y \in L$ such that $x - y$ is orthogonal to ξ. It follows from the construction of ξ that $x - y \in \eta^\perp$.

Hence, for all $x \in K + v$, there exists $y \in L$ such that $x - y \in \eta^\perp$. This implies that $K_\eta + v_\eta \subseteq L_\eta$.

We have shown that L_η contains a translate of K_η for each i-subspace η of $\mathbb{R}^{j+1}$ such that $\dim(\eta^\perp \cap \mathbb{R}^j) = j - i$. Since this is a dense family of i-subspaces, the lemma follows more generally for all i-subspaces of $\mathbb{R}^{j+1}$. By a suitable iteration of this argument, the lemma then follows for i-subspaces of $\mathbb{R}^n$, for any $n > j$. $\square$

We can now generalize the counterexample of Section 2.

Theorem 3.2. *Let $n \geqslant 3$ and $1 \leqslant k < n$. There exist convex bodies $K, L \in \mathscr{K}_n$ such that L_ξ contains a translate of K_ξ for each k-dimensional subspace ξ, while $V_m(K) > V_m(L)$ for all $m > k$.*

Note that this theorem is already well known for the case $k = 1$. In that particular case, the covering condition merely asserts that the width of K in any direction is smaller than or equal to the corresponding width of L. The novel aspect of this result addresses the cases in which $2 \leqslant k < n$.

Proof. If $k = 1$, then let K be an n-simplex, and let $M = \frac{1}{2}K + (-\frac{1}{2}K)$, the central symmetral of K. It then follows from the Minkowski mixed volume inequality (4) and the classical mean projection formulas for intrinsic volumes [5, p. 408], [12, p. 125], [19, p. 235], that $V_m(M) > V_m(K)$ for $m \geqslant 2$, while K and M have identical width in every direction.

Now suppose that $k \geqslant 2$. Let $\hat{K}$ and $\hat{L}$ be chosen in $\mathscr{K}_{k+1}$ so that $\hat{L}_\xi$ contains a translate of $\hat{K}_\xi$ for each k-subspace ξ of $\mathbb{R}^{k+1}$, while $V_{k+1}(\hat{K}) > V_{k+1}(\hat{L})$. (One could follow the explicit construction given in Section 2, for example.)

If $n = k + 1$, we are done. If $n > k + 1$, embed $\hat{K}$ and $\hat{L}$ in $\mathbb{R}^n$ via the usual coordinate embedding of $\mathbb{R}^{k+1}$ in $\mathbb{R}^n$. Then $\hat{L}_\xi$ can cover $\hat{K}_\xi$ for all k-subspaces ξ of $\mathbb{R}^n$, by Lemma 3.1.

Let C denote the unit cube in $\mathbb{R}^{n-k-1}$ with edges parallel to the standard axes in the orthogonal complement to $\mathbb{R}^{k+1}$ in $\mathbb{R}^n$. Let $K = \hat{K} + \epsilon C$ and $L = \hat{L} + \epsilon C$. Then L_ξ contains a translate of K_ξ for each k-dimensional subspace ξ once again, since $K_\xi = \hat{K}_\xi + \epsilon C_\xi$, and similarly for L.

Moreover, if $m \geqslant k + 1$ then

$$V_m(K) = V_m(\hat{K} + \epsilon C) = \sum_{i+j=m} V_i(\hat{K}) V_j(C)\epsilon^j$$

by the Cartesian product formula for intrinsic volumes [12, p. 130]. Hence,

$$\begin{aligned} V_m(K) &= \sum_{i=0}^{k+1} V_i(\hat{K}) V_{m-i}(C)\epsilon^{m-i} \\ &= \sum_{i=0}^{k+1} \binom{n-k-1}{m-i} V_i(\hat{K})\epsilon^{m-i} \\ &= \epsilon^{m-k-1}\binom{n-k-1}{m-k-1} V_{k+1}(\hat{K}) + f_K(\epsilon) \end{aligned}$$

where $f_K(\epsilon)$ is a polynomial in ϵ composed of monomials having degree greater than $m-k-1$. A similar formula holds for $V_m(L)$. Therefore,

$$V_m(K) - V_m(L) = \epsilon^{m-k-1}\binom{n-k-1}{m-k-1}\big(V_{k+1}(\hat{K}) - V_{k+1}(\hat{L})\big) + \big(f_K(\epsilon) - f_L(\epsilon)\big),$$

where $f_K(\epsilon) - f_L(\epsilon)$ is a polynomial in ϵ composed of monomials having degree greater than $m-k-1$. Since the lowest degree coefficient of the polynomial formula for $V_m(K) - V_m(L)$ is positive, we have $V_m(K) - V_m(L) > 0$ when $\epsilon > 0$ is sufficiently small. □

4. Cylinders and shadow covering

Let $K \in \mathscr{K}_n$ and suppose that $P \in \mathscr{K}_n$ is a polytope with non-empty interior. We say that P *circumscribes* K if $K \subseteq P$ and K also meets every facet of P.

Lemma 4.1 *(Circumscribing Lemma). Let $K, P \in \mathscr{K}_n$, where P is a polytope with non-empty interior. If P circumscribes K then*

$$V_{n-1,1}(P, K) = V_n(P). \tag{14}$$

If we are only given that $K \subseteq P$, then (14) *holds if and only if P circumscribes K.*

Proof. If $K \subseteq P$ and if K meets every facet of P, then $h_K(u) = h_P(u)$ whenever the direction u is normal to a facet of P. Since P is a polytope, the mixed volume formula (2) yields

$$V_{n-1,1}(P, K) = \frac{1}{n}\sum_{u \perp \partial P} h_K(u) V_{n-1}\big(P^u\big) = \frac{1}{n}\sum_{u \perp \partial P} h_P(u) V_{n-1}\big(P^u\big) = V_n(P).$$

Conversely, if we are only given that $K \subseteq P$, then $h_K(u) \leqslant h_P(u)$, with equality for all facet normals u if and only if P circumscribes K, so that (14) holds if and only if P circumscribes K. □

The case in which P is a simplex is especially important, because of the following theorem of Lutwak [15] (see also [12, p. 54]), itself a consequence of Helly's theorem.

Theorem 4.2 *(Lutwak's Containment Theorem). Let $K, L \in \mathscr{K}_n$. Suppose that, for every simplex Δ such that $L \subseteq \Delta$, there is a vector $v_\Delta \in \mathbb{R}^n$ such that $K + v_\Delta \subseteq \Delta$. Then there is a vector $v \in \mathbb{R}^n$ such that $K + v \subseteq L$.*

Lutwak's theorem combines with the Circumscribing Lemma 4.1 to yield the following useful corollary (also from [15]).

Corollary 4.3. *Let $K, L \in \mathscr{K}_n$. The inequality*

$$V_{n-1,1}(\Delta, K) \leqslant V_{n-1,1}(\Delta, L)$$

holds for all simplices Δ, if and only if there exists $v \in \mathbb{R}^n$ such that $K + v \subseteq L$.

Suppose that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m > 0$ are positive integers such that $\lambda_1 + \lambda_2 + \cdots + \lambda_m = n$. Denote $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$. The vector λ is sometimes called a *partition* of the positive integer n. Using this notation, the size of the largest part of any partition λ is given by the first entry λ_1.

A convex body $K \in \mathscr{K}_n$ will be called λ-*decomposable* if there exist affine subspaces ξ_i of $\mathbb{R}^n$ such that $\dim \xi_i = \lambda_i$ and $\mathbb{R}^n = \xi_1 \oplus \cdots \oplus \xi_m$, and if there exist compact convex sets $K_i \subseteq \xi_i$ such that $K = K_1 + \cdots + K_m$. In this case we will write $K = K_1 \oplus \cdots \oplus K_m$. The body K will be called λ-*ortho-decomposable* if $\xi_i \perp \xi_j$ for each $i \neq j$.

For example, a cylinder (prism) is an $(n-1,1)$-decomposable body. A $(1,1,\ldots,1)$-decomposable body is a parallelotope, while a $(1,1,\ldots,1)$-ortho-decomposable body is an orthogonal box.

For $k \in \{1,\ldots,n-1\}$, denote by $G(n,k)$ the collection of all k-dimensional linear subspaces ξ of $\mathbb{R}^n$, sometimes called the (n,k)-Grassmannian. For $\xi \in G(n,k)$ and $K \in \mathscr{K}_n$, we continue to denote by K_ξ the orthogonal projection of the body K onto the subspace ξ.

Lutwak's Containment Theorem 4.2 and its Corollary 4.3 lead to the following useful condition for determining when the shadows of one body can cover those of another.

Theorem 4.4 *(First Shadow Containment Theorem). Let $K, L \in \mathscr{K}_n$, and suppose that $1 \leqslant k \leqslant n-1$. The orthogonal projection L_ξ contains a translate of the corresponding projection K_ξ for each $\xi \in G(n,k)$ if and only if*

$$V_{n-1,1}(C, K) \leqslant V_{n-1,1}(C, L)$$

for all λ-ortho-decomposable $C \in \mathscr{K}_n$ such that $\lambda_1 \leqslant k$.

Proof. Suppose that C is a λ-ortho-decomposable polytope, with orthogonal decomposition $C = a_1C_1 \oplus \cdots \oplus a_mC_m$, where each C_i has affine hull parallel to a subspace ξ_i of dimension λ_i and $a_1, \ldots, a_m > 0$. Note that

$$V_n(C) = V_{\lambda_1}(a_1C_1) \cdots V_{\lambda_m}(a_mC_m) = a_1^{\lambda_1} \cdots a_m^{\lambda_m} V_{\lambda_1}(C_1) \cdots V_{\lambda_m}(C_m),$$

since the decomposition is orthogonal.

It follows from (2) that

$$V_{n-1,1}(C, K) = \frac{1}{n} \sum_{u \perp \partial C} h_K(u) V_{n-1}\bigl(C^u\bigr)$$

where the sum is taken over all unit directions $u \in \mathbb{R}^n$ normal to facets of C. The product structure of C then implies that

$$V_{n-1,1}(C, K) = \frac{1}{n} \sum_{i=1}^{m} \sum_{u \perp \partial C_i} h_K(u) V_{\lambda_1}(a_1C_1) \cdots V_{\lambda_i-1}\bigl(a_iC_i^u\bigr) \cdots V_{\lambda_m}(a_mC_m)$$

where, for each i, the inner sum is taken over all unit directions $u \in \xi_i$ normal to facets of C_i. Hence,

$$\begin{aligned} V_{n-1,1}(C, K) &= \frac{1}{n} \sum_{i=1}^{m} \frac{V_n(C)}{V_{\lambda_i}(a_iC_i)} \sum_{u \perp \partial C_i} h_K(u) V_{\lambda_i-1}\bigl(C_i^u\bigr) a_i^{\lambda_i-1} \\ &= \frac{1}{n} \sum_{i=1}^{m} \frac{\lambda_i V_n(C)}{a_i V_{\lambda_i}(C_i)} V_{\lambda_i-1,1}(C_i, K_{\xi_i}) \end{aligned} \tag{15}$$

for all $a_1, \ldots, a_m > 0$.

If L_ξ contains a translate of K_ξ for each $\xi \in G(n,k)$, then L_η contains a translate of K_η for all lower-dimensional subspaces $\eta \in G(n, j)$, where $1 \leqslant j \leqslant k$. In particular L_{ξ_i} can cover K_{ξ_i} for each i, since $\dim \xi_i = \lambda_i \leqslant \lambda_1 \leqslant k$. It follows that each $V_{i-1,1}(C_i, K_{\xi_i}) \leqslant V_{i-1,1}(C_i, L_{\xi_i})$ by the monotonicity and translation invariance of mixed volumes. The identity (15) now implies that $V_{n-1,1}(C, K) \leqslant V_{n-1,1}(C, L)$ for all λ-ortho-decomposable polytopes C. This inequality then holds for arbitrary λ-ortho-decomposable bodies C by continuity of mixed volumes.

Conversely, suppose that $V_{n-1,1}(C, K) \leqslant V_{n-1,1}(C, L)$ for all λ-ortho-decomposable $C \in \mathscr{K}_n$ such that $\lambda_1 \leqslant k$.

Suppose that $\xi \in G(n,i)$ for some $i \leqslant k$. If Δ is an i-simplex in $\xi \in G(n,i)$, let $C = \Delta + \epsilon Z$, where $\epsilon > 0$, and where Z is a unit cube in $\xi^\perp$. The facets of C are either products of Δ with facets of the cube ϵZ or products of ϵZ with facets of Δ. As in (15), it follows that

$$V_{n-1,1}(C, K) = \frac{i}{n} V_{i-1,1}(\Delta, K_\xi)\epsilon^{n-i} + \frac{1}{n} V_i(\Delta)\Biggl(\sum_{v \perp \partial Z} h_K(v)\Biggr)\epsilon^{n-i-1},$$

where the last sum is taken over unit directions v normal to the facets of Z in $\xi^\perp$. A similar expression holds for $V_{n-1,1}(C, L)$. Since $V_{n-1,1}(C, K) \leqslant V_{n-1,1}(C, L)$ for every value of $\epsilon > 0$, we have $V_{i-1,1}(\Delta, K_\xi) \leqslant V_{i-1,1}(\Delta, L_\xi)$.

It follows that $V_{\lambda_i-1,1}(\Delta, K_{\xi_i}) \leqslant V_{\lambda_i-1,1}(\Delta, L_{\xi_i})$ for every λ_i-simplex Δ in every λ_i-dimensional subspace ξ_i of $\mathbb{R}^n$, so that each L_{ξ_i} contains a translate of K_{ξ_i} by Corollary 4.3. □

5. Cylinder bodies and shadow covering

So far we have restricted attention to *orthogonal* cylinders and decomposable sets. However, the previous results generalize easily to arbitrary (possibly oblique) cylinders and decompositions.

For $S \subseteq \mathbb{R}^n$ and a nonzero vector u, let $\mathcal{L}_S(u)$ denote the set of straight lines in $\mathbb{R}^n$ parallel to u and meeting the set S.

Proposition 5.1. *Let $K, L \in \mathscr{K}_n$. Let $\psi : \mathbb{R}^n \to \mathbb{R}^n$ be a non-singular linear transformation. Then L_u contains a translate of K_u for all unit directions u if and only if $(\psi L)_u$ contains a translate of $(\psi K)_u$ for all u.*

Proof. The projection L_u contains a translate of K_u for each unit vector u if and only if, for each u, there exists v_u such that

$$\mathcal{L}_{K+v_u}(u) \subseteq \mathcal{L}_L(u). \tag{16}$$

But $\mathcal{L}_{K+v_u}(u) = \mathcal{L}_K(u) + v_u$ and $\psi\mathcal{L}_K(u) = \mathcal{L}_{\psi K}(\psi u)$. It follows that (16) holds if and only if $\mathcal{L}_K(u) + v_u \subseteq \mathcal{L}_L(u)$, if and only if

$$\mathcal{L}_{\psi K}(\psi u) + \psi v_u \subseteq \mathcal{L}_{\psi L}(\psi u) \quad \text{for all unit } u.$$

Set

$$\tilde{u} = \frac{\psi u}{|\psi u|} \quad \text{and} \quad \tilde{v} = \psi v_u.$$

The relation (16) now holds if and only if, for all $\tilde{u}$, there exists $\tilde{v}$ such that

$$\mathcal{L}_{\psi K}(\tilde{u}) + \tilde{v} \subseteq \mathcal{L}_{\psi L}(\tilde{u}),$$

which holds if and only if $(\psi L)_{\tilde{u}}$ contains a translate of $(\psi K)_{\tilde{u}}$ for all $\tilde{u}$. □

We are now in a position to define a much larger family of objects that serve to generalize the Shadow Containment Theorem 4.4.

Definition 5.2. For each $k \in \{1, \ldots, n\}$ denote by $\mathcal{C}_{n,k}$ the set of all bodies $K \in \mathscr{K}_n$ that can be approximated (with respect to the usual Hausdorff metric) by Blaschke combinations of λ-decomposable sets for any λ such that $\lambda_1 \leqslant k$. Elements of $\mathcal{C}_{n,k}$ will be called the k-cylinder bodies of $\mathbb{R}^n$.

Recall that any centrally symmetric polytope with interior is a Blaschke sum of parallelotopes. It follows that $\mathcal{C}_{n,1}$ is precisely the set of all centrally symmetric convex bodies in $\mathbb{R}^n$. For $n \geqslant 3$ and $k \geqslant 2$, the cylinder bodies $\mathcal{C}_{n,k}$ are a larger family of objects. For example, a triangular prism in $\mathbb{R}^3$ lies in $\mathcal{C}_{3,2}$, but not in $\mathcal{C}_{3,1}$, since it is not centrally symmetric. Note also that $\mathcal{C}_{n,k}$ is closed under affine transformations.

The definition of $\mathcal{C}_{n,k}$ depends on the ambient dimension n as well as the value k, because the notion of Blaschke sum # depends on n. For example, while Minkowski sum satisfies the projection identity $(K + L)_\xi = K_\xi + L_\xi$ for subspaces $\xi \subseteq \mathbb{R}^n$, the analogous statement need not hold for Blaschke summation.

Note also that $\mathcal{C}_{n,n} = \mathscr{K}_n$ by definition. Moreover, it follows from the definition that $\mathcal{C}_{n,i} \subseteq \mathcal{C}_{n,j}$ whenever $i \leqslant j$. It will be shown in Section 6 that $\mathcal{C}_{n,i}$ is a *proper* subset of $\mathcal{C}_{n,j}$ when $i < j$. In particular, it will be seen that full-dimensional simplices are *not* k-cylinder bodies of $\mathbb{R}^n$ for any $k < n$. A necessary condition for being a k-cylinder body will be described in Section 7.

The significance of each collection $\mathcal{C}_{n,k}$ is described in part by the following theorem.

Theorem 5.3 *(Second Shadow Containment Theorem). Let $K, L \in \mathscr{K}_n$ and let $1 \leqslant k \leqslant n$. The following are equivalent:*

(i) *The orthogonal projection L_ξ of L contains a translate of the corresponding projection K_ξ of K for each subspace $\xi \in G(n, k)$.*
(ii) *The affine projection πL of L contains a translate of the corresponding projection πK of K for each affine projection π of rank k.*
(iii) $V_{n-1,1}(C, K) \leqslant V_{n-1,1}(C, L)$ *for all* λ-ortho-decomposable *sets C such that $\lambda_1 \leqslant k$.*
(iv) $V_{n-1,1}(C, K) \leqslant V_{n-1,1}(C, L)$ *for all* λ-decomposable *sets C such that $\lambda_1 \leqslant k$.*
(v) $V_{n-1,1}(Q, K) \leqslant V_{n-1,1}(Q, L)$ *for all k-cylinder bodies $Q \in \mathcal{C}_{n,k}$.*

Proof. The equivalence of (i) and (ii) follows from Proposition 5.1. The equivalence of (i) and (iii) follows from Theorem 4.4.

To show that (iii) implies (iv), suppose that (iii) holds for the pair K, L. It follows from (i) and Proposition 5.1 that (i) also holds for the pair of bodies $\psi^{-1}K, \psi^{-1}L$, for any non-singular affine transformation ψ. Therefore (iii) also holds for the pair of bodies $\psi^{-1}K, \psi^{-1}L$; that is,

$$V_{n-1,1}\bigl(C, \psi^{-1}K\bigr) \leqslant V_{n-1,1}\bigl(C, \psi^{-1}L\bigr)$$

for all λ-ortho-decomposable sets C such that $\lambda_1 \leqslant k$. Let us suppose that ψ has unit determinant. Then $V_{n-1,1}(\psi C, K) = V_{n-1,1}(C, \psi^{-1}K)$, and similarly for L, by the affine invariance of (mixed) volumes, so that

$$V_{n-1,1}(\psi C, K) \leqslant V_{n-1,1}(\psi C, L)$$

for all λ-ortho-decomposable sets C such that $\lambda_1 \leqslant k$. If C' is a λ-decomposable set, then $C' = \psi C$ for some λ-ortho-decomposable set C and some affine transformation ψ of unit determinant. Statement (iv) now follows.

(iv) implies (v) by the Blaschke-linearity of the functional $V_{n-1,1}(\cdot,\cdot)$ in its *first* parameter and the continuity of $V_{n-1,1}$.

Finally, (v) implies (iv), and (iv) implies (iii), in both cases *a fortiori*. □

6. A positive answer for covering cylinder bodies

In Section 3 we described examples of convex bodies K and L such that the orthogonal projection L_ξ contains a translate of the corresponding projection K_ξ for each $\xi \in G(n, k)$, even though $V_n(L) < V_n(K)$. The next theorem shows that this volume anomaly can be avoided if $L \in \mathcal{C}_{n,k}$.

Theorem 6.1. *Let $K, L \in \mathcal{K}_n$ and let $1 \leqslant k \leqslant n-1$. Suppose that the orthogonal projection L_ξ contains a translate of the corresponding projection K_ξ for each $\xi \in G(n,k)$. If $L \in \mathcal{C}_{n,k}$, then $V_n(K) \leqslant V_n(L)$.*

If, in addition, the set L has non-empty interior, then $V_n(K) = V_n(L)$ if and only if K and L are translates.

Proof. If the orthogonal projection L_ξ of L contains a translate of the corresponding projection K_ξ of K for each $\xi \in G(n,k)$, then

$$V_{n-1,1}(Q, K) \leqslant V_{n-1,1}(Q, L)$$

for all $Q \in \mathcal{C}_{n,k}$, by Theorem 5.3. If $L \in \mathcal{C}_{n,k}$ as well, then

$$V_{n-1,1}(L, K) \leqslant V_{n-1,1}(L, L) = V_n(L).$$

Meanwhile, the Minkowski mixed volume inequality (4) asserts that

$$V_n(L)^{(n-1)/n} V_n(K)^{1/n} \leqslant V_{n-1,1}(L, K).$$

Hence $V_n(K) \leqslant V_n(L)$. If equality holds and $V_n(L) > 0$, then K and L are homothetic bodies of the same volume by the equality conditions of (4), so that K and L must be translates. $\square$

The simplicial counterexamples of Section 2 along with Theorem 6.1 yield the following immediate corollary.

Corollary 6.2. *An n-dimensional simplex is never an element of $\mathcal{C}_{n,n-1}$.*

In particular, the collection of $(n-1)$-cylinder bodies $\mathcal{C}_{n,n-1}$ forms a *proper* subset of $\mathcal{C}_{n,n} = \mathcal{K}_n$.

More generally we have the following.

Corollary 6.3. *For $1 \leqslant i < j \leqslant n$ the set $\mathcal{C}_{n,i}$ is a proper subset of $\mathcal{C}_{n,j}$.*

Proof. It follows directly from the definition of $\mathcal{C}_{n,i}$ that $\mathcal{C}_{n,i} \subseteq \mathcal{C}_{n,j}$ when $i < j$. It remains to show that $\mathcal{C}_{n,i} \neq \mathcal{C}_{n,j}$ when $i < j$.

To see this, observe that the set L constructed in the proof of Theorem 3.2 satisfies $L \in \mathcal{C}_{n,k+1}$, because L is λ-decomposable for $\lambda = (k+1, 1, \ldots, 1)$. Let K also be chosen as in the proof of Theorem 3.2. Recall that the k-shadow L_ξ contains a translate of K_ξ for every $\xi \in G(n,k)$. Since $V_n(L) < V_n(K)$, it follows from Theorem 6.1 that $L \notin \mathcal{C}_{n,k}$. Hence $\mathcal{C}_{n,k} \neq \mathcal{C}_{n,k+1}$. $\square$

In other words, the collections $\mathcal{C}_{n,k}$ form a *strictly* increasing chain

$$\mathcal{C}_{n,1} \subset \mathcal{C}_{n,2} \subset \cdots \subset \mathcal{C}_{n,n-1} \subset \mathcal{C}_{n,n} = \mathcal{K}_n$$

where the elements of $\mathcal{C}_{n,1}$ are precisely the centrally symmetric sets in $\mathcal{K}_n$.

7. A geometric inequality for cylinder bodies

For positive integers $n \geqslant 2$ denote

$$\sigma_n = \begin{cases} \frac{\sqrt{n+2}}{2n+2} & \text{if } n \text{ is even,} \\ \frac{1}{2\sqrt{n}} & \text{if } n \text{ is odd.} \end{cases}$$

Recall that we denote the surface area of a convex body K by $S(K)$ and the minimal width of K by d_K.

Theorem 7.1 *(Cylinder body inequality). Let $K \in \mathscr{K}_n$. If $K \in \mathcal{C}_{n,i}$, then*

$$\sigma_i d_K S(K) \leqslant n V_n(K). \tag{17}$$

Proof. If $V_n(K) = 0$ then $d_K = 0$ as well, so that both sides of (17) are zero.

Suppose that $V_n(K) > 0$. By Steinhagen's inequality (10) and the fact that $\dim K = n$,

$$r_{K_\xi} \geqslant \sigma_i d_{K_\xi} \geqslant \sigma_i d_K,$$

for each subspace $\xi \in G(n,i)$, where d_{K_ξ} is computed from within the subspace ξ.

For $0 \leqslant \epsilon \leqslant 1$, denote $K^\epsilon = \epsilon K + (1-\epsilon)\sigma_i d_K B_n$, where B_n is an n-dimensional unit Euclidean ball. Since

$$\sigma_i d_K (B_n)_\xi \subseteq r_{K_\xi}(B_n)_\xi \subseteq K_\xi \text{ up to translation,}$$

we have

$$K_\xi^\epsilon \subseteq \epsilon K_\xi + (1-\epsilon)K_\xi = K_\xi \text{ up to translation,}$$

for each subspace $\xi \in G(n,i)$. If $K \in \mathcal{C}_{n,i}$, then $V_n(K^\epsilon) \leqslant V_n(K)$, by Theorem 6.1. Moreover, Steiner's formula (12) implies that

$$V_n\big(K^\epsilon\big) = \epsilon^n V_n(K) + \epsilon^{n-1}(1-\epsilon)\sigma_i d_K S(K) + (1-\epsilon)^2 f(\epsilon),$$

where $f(\epsilon)$ is a polynomial in ϵ. Since $V_n(K^\epsilon) - V_n(K) \leqslant 0$ for $0 \leqslant \epsilon < 1$, we have

$$\big(\epsilon^n - 1\big)V_n(K) + \epsilon^{n-1}(1-\epsilon)\sigma_i d_K S(K) + (1-\epsilon)^2 f(\epsilon) \leqslant 0$$

so that

$$\epsilon^{n-1}\sigma_i d_K S(K) + (1-\epsilon) f(\epsilon) \leqslant \big(1 + \epsilon + \cdots + \epsilon^{n-1}\big) V_n(K)$$

for all $0 \leqslant \epsilon < 1$. As $\epsilon \to 1$ this implies that $\sigma_i d_K S(K) \leqslant n V_n(K)$. $\square$

8. A volume ratio bound

In Section 2 we described n-dimensional convex bodies K and L such that the orthogonal projection L_u contains a translate of K_u for every direction u, while $V_n(K) > V_n(L)$. In such instances, one could ask instead for an upper bound on the volume ratio $\frac{V_n(K)}{V_n(L)}$. An application of Theorem 6.1 yields the following crude estimate.

Theorem 8.1. *Let $K, L \in \mathcal{K}_n$, and suppose that the orthogonal $(n-1)$-dimensional projection L_u contains a translate of the corresponding projection K_u for each direction u. Then $V_n(K) \leqslant nV_n(L)$.*

Recall that the *diameter* D_K of a convex body K is the maximum distance between any two points of the body K, and is also equal to the maximum width, that is, the maximum distance between any two parallel supporting hyperplanes of K.

Proof. Suppose that the diameter D_L of L is realized in the unit direction v. A standard Steiner symmetrization (or, alternatively, shaking) argument implies that

$$V_n(L) \geqslant \frac{1}{n} D_L V_{n-1}(L_v).$$

Let $\bar{v}$ denote the unit line segment having endpoints at the origin o and at v, and let C be the orthogonal cylinder in $\mathbb{R}^n$ given by $C = L_v \oplus D_L \bar{v}$. After a suitable translation, we may assume that $L \subseteq C$. From the original covering assumption for L it then follows that each projection K_u can be translated inside the corresponding projection C_u of the cylinder C. By Theorem 6.1, it then follows that

$$V_n(K) \leqslant V_n(C) = D_L V_{n-1}(L_v) \leqslant nV_n(L). \qquad \square$$

9. Some open questions

The results of the previous sections motivate several open questions about convex bodies and projections.

I. Let $K, L \in \mathcal{K}_n$ such that $V_n(L) > 0$, and let $1 \leqslant k \leqslant n-1$. Suppose that the orthogonal projection L_ξ contains a translate of the corresponding projection K_ξ of K for each subspace $\xi \in G(n,k)$.
What is the best upper bound for the ratio $\frac{V_n(K)}{V_n(L)}$?
An answer to this question would improve Theorem 8.1.

II. Given a partition λ of a positive integer n, define $\mathcal{C}_\lambda$ to be the collection of all convex bodies that can be approximated by Blaschke sums of μ-decomposable convex bodies, taken over all partitions μ that refine the partition λ.
If λ and σ are incomparable partitions of n (with respect to partition refinement), how are $\mathcal{C}_\lambda$ and $\mathcal{C}_\sigma$ related? Can we describe their relative geometric significance in the context of projections?

III. Up to translation, zonoids can be thought of as the image of the projection body operator on convex sets or of the cosine transform on support functions, and intersection bodies are constructed by taking the Radon transform of the radial function of a convex (or star-shaped) set [5, p. 323], [14], [19, p. 416]. Is there an analogous integral geometric description for the families $\mathcal{C}_\lambda$ and $\mathcal{C}_{n,k}$?

IV. What simple tests, conditions, or inequalities determine whether or not a convex body K is an element of some $\mathcal{C}_\lambda$ or $\mathcal{C}_{n,k}$?

V. Let $K, L \in \mathscr{K}_n$ such that $V_n(L) > 0$, and let $1 \leqslant k \leqslant n-1$. Suppose that the orthogonal projection L_ξ contains a translate of the corresponding projection K_ξ for each $\xi \in G(n,k)$. Under what simple (easy to state, easy to verify) additional conditions does it follow that K can be translated inside L?
Partial answers to Question V are given in [10,11].

VI. Let $K, L \in \mathscr{K}_n$ such that $V_n(L) > 0$, and let $1 \leqslant k \leqslant n-1$. Suppose that the orthogonal projection L_ξ contains a *congruent copy* of the corresponding projection K_ξ (under some rigid motion) for each $\xi \in G(n,k)$.
Under what simple (easy to state, easy to verify) additional conditions does it follow that L contains a congruent copy of K?

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