ADVANCES IN MATHEMATICS **121**, 80–101 (1996)
ARTICLE NO. 0048

Star Valuations and Dual Mixed Volumes

DANIEL A. KLAIN*

School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332-0160

Received January 1, 1994; accepted Februay 2, 1994

INTRODUCTION

Since its creation by Brunn and Minkowski, what has become known as the Brunn–Minkowski theory has provided powerful machinery to solve a broad variety of inverse problems with stereological data. The machinery of the Brunn–Minkowski theory includes mixed volumes (of Minkowski), symmetrization techniques (such as those of Steiner and Blaschke), isoperimetric inequalities (such as the Brunn–Minkowski, Minkowski, and Aleksandrov–Fenchel inequalities), integral transforms (such as the cosine transform), and important auxiliary bodies associated with these transforms (such as Minkowski's projection bodies). Schneider's recent book [22] on the Brunn–Minkowski theory is the best available introduction to the subject.

While the Brunn–Minkowski theory has proven to be of enormous value in answering inverse questions regarding projections of convex bodies onto subspaces, the theory has been of little value in answering inverse questions with data regarding *intersections* with subspaces. However, recent advancements have been made in the development of a *dual* Brunn–Minkowski theory [3, 4, 5, 6, 8, 9, 11, 13, 16, 23, 26, 27, 28, 29] which has been tailored specifically for dealing with such questions. In contrast to the Brunn–Minkowski theory, in the dual theory convex bodies are replaced by star-shaped sets, and projections onto subspaces are replaced by intersections with subspaces. The machinery of the dual theory includes dual mixed volumes (introduced by Lutwak [15, 16]), dual isoperimetric inequalities ([16]), and important auxiliary bodies known as intersection bodies (first introduced by Busemann for special centered convex bodies in [1]; later defined for star-shaped sets by Lutwak [16]. See also [9]). A comprehensive introduction to geometric tomography, including the dual Brunn–Minkowski theory, may be found in Gardner's book [17].

Unfortunately, there still remain fundamental and foundational problems with the dual Brunn–Minkowski theory. One of the most beautiful and

* E-mail: klain@math.gatech.edu.

0001-8708/96 $18.00

important results of twentieth century convexity is Hadwiger's characterization theorem for the elementary mixed volumes (Quermassintegrals) (See [12, 14]). Hadwiger's characterization leads to effortless proofs of numerous results in integral geometry, including the mean projection formulas for convex bodies [22] and various kinematic formulas [20, 24]. This result also provides a connection between rigid motion invariant set functions and symmetric polynomials [2, 20]. Hadwiger's theorem is of such fundamental importance that any candidate for a dual theory must possess a dual analogue of this theorem. As will be seen below, the dual Brunn–Minkowski theory, as currently understood, is not sufficiently rich to be able to accommodate a dual of Hadwiger's theorem.

The purpose of this paper is two-fold. First, it will be shown that the natural setting for a dual Brunn–Minkowski theory is larger than that envisioned by previous investigators. In the sections that follow I present an extension of the dual Brunn–Minkowski theory to a broad class of star-shaped sets previously inaccessible to the dual theory.

Second, I present a classification theorem for homogeneous valuations on star-shaped sets. Included in this result is a classification of continuous and homogeneous valuations that are invariant under the action of rotations. This result leads in turn to a characterization theorem for the dual elementary mixed volumes (dual Quermassintegrals), a dual analogue to Hadwiger's characterization of the elementary Minkowski mixed volumes.

Backround material is sketched, and proofs are given for the main results. For a more detailed treatment, see [13].

Section 1 contains a summary of certain results from geometric convexity, the star analogues of which are developed in the subsequent sections. Of particular importance is the unusual definition of the Hausdorff topology on the set of convex bodies, of which the dual analogue proves a crucial tool for understanding valuations on the class of star-shaped sets.

Section 2 begins with a definition for "L^n-stars", the class of bodies with which we work throughout. This is followed by a new definition for the dual Hausdorff topology on L^n-stars (corresponding to the L^n topology on radial functions). Unlike previous definitions, this new analogue to the Hausdorff topology permits a *continuous* action of the rotation group $SO(n)$ on the space of L^n-stars. Extensions are given for the dual mixed volumes to this larger class of star-shaped sets. Definitions are given for valuations (also called *star valuations*) of L^n-stars, and inclusion-exclusion properties of dual mixed volumes are worked out.

The main result of this paper, presented in Section 4, is the classification of all continuous star valuations that are homogeneous with respect to dilation. This classification leads in turn to a characterization theorem for dual mixed volumes of pairs of L^n-stars. These results are shown to be analogues of classical theorems of McMullen [18] and Goodey and Weil [10] regarding homogeneous valuations of *convex* sets. I also present a dual Hadwiger theorem for homogeneous valuations that are rotation invariant.

1. CONVEXITY

In this section we summarize results from the classical convexity theory. Important modifications will be made in several of the classical definitions. For detailed backround material, see [22].

We shall denote n-dimensional Euclidean space by $\mathbf{R}^n$. The spherical Lebesgue measure on the $(n-1)$-dimensional unit sphere $\mathbf{S}^{n-1}$ shall be denoted by S. For a function $f: \mathbf{S}^{n-1} \to R$ that is measurable with respect to S, let

$$\|f\|_p = \left(\int_{\mathbf{S}^{n-1}} |f|^p dS \right)^{1/p}.$$

A measurable function f on $\mathbf{S}^{n-1}$ is called L^p-integrable, or simply L^p, if $\|f\|_p < \infty$.

Let $\mathcal{K}^n$ denote the set of all convex bodies in $\mathbf{R}^n$; i.e. the set of all compact convex subsets of $\mathbf{R}^n$. A convex body $K \in \mathcal{K}^n$ is determined uniquely by its *support function* $h_K: \mathbf{S}^{n-1} \to \mathbf{R}$, defined by $h_K(u) = \max_{x \in K}\{x \cdot u\}$, where $\cdot$ denotes the standard inner product on $\mathbf{R}^n$. The support function h_K is a continuous function on the unit sphere. For $K, L \in \mathcal{K}^n$, we have $K \subseteq L$ if and only if $h_K \leqslant h_L$.

DEFINITION 1.1. Given $K_1, K_2, ..., K_m \in \mathcal{K}^n$, and positive real numbers $\lambda_1, \lambda_2, ..., \lambda_m$, the *Minkowski linear combination* $K = \lambda_1 K_1 + \lambda_2 K_2 + \cdots + \lambda_m K_m$ is the convex body whose support function is given by $h_K = \sum_{j=1}^m \lambda_j h_{K_j}$.

It is not hard to show that K consists of all vector sums $\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_m x_m$ of points $x_j \in K_j$.

DEFINITION 1.2. Let $K_1, K_2, K_3, ... \in \mathcal{K}^n$. The sequence $\{K_j\}_1^\infty$ *converges* to the convex body K in the *Hausdorff topology* if $\|h_{K_j} - h_K\|_n \to 0$ as $j \to \infty$.

This definition differs from the usual description of the Hausdorff topology on $\mathcal{K}^n$, in which the support functions of a convergent sequence of bodies must be uniformly convergent rather than convergent in L^n. However, it follows from a result of Vitale [25] that a sequence of support functions converges in the L^n topology if and only if the sequence converges uniformly. Hence the two definitions are equivalent.

The volume (Lebesgue measure) of a convex body K is denoted by $V(K)$. For computing the volume of a Minkowski linear combination, we have the following well-known result [22, p. 275]. Let $[m]$ denote the set of natural numbers 1, 2, ..., m.

THEOREM 1.3. *If* $K_1, ..., K_m \in \mathscr{K}^n$ *and* $\lambda_1, ..., \lambda_m > 0$, *then*

$$V(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{i_1, ..., i_n \in [m]} V(K_{i_1}, ..., K_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n},$$

where each coefficient $V(K_{i_1}, ..., K_{i_n})$ *depends only on the bodies* $K_{i_1}, ..., K_{i_n}$.

Given $K_1, ..., K_n \in \mathscr{K}^n$, the coefficient $V(K_1, ..., K_n)$ is called the *Minkowski mixed volume* of the convex bodies $K_1, ..., K_n$. It is well-known, but not trivial, that the mixed volume $V(K_1, ..., K_n)$ is a non-negative continuous symmetric function in n variables on the set $\mathscr{K}^n$ and is monotonic with respect to the subset partial ordering on $\mathscr{K}^n$.

For $0 \leqslant i \leqslant n$ and $K, L \in \mathscr{K}^n$, define $V_i(K, L) = V(K, ..., K, L, ..., L)$, where K appears $n-i$ times and L appears i times in the right-hand expression. Important special cases of the Minkowski mixed volumes are the *elementary mixed volumes* (or *quermassintegrals*) of a convex body K, defined by $W_i(K) = V_i(K, B)$, for $0 \leqslant i \leqslant n$, where B denotes the unit ball in $\mathbf{R}^n$ centered at the origin.

The elementary mixed volumes are also known as the *mean projection measures*. Let v_i denote the i-dimensional volume on $\mathbf{R}^i$, and let $\mathrm{Gr}(n, i)$ denote the Grassmannian of i-dimensional subspaces of $\mathbf{R}^n$. For $\xi \in \mathrm{Gr}(n, i)$ and $K \in \mathscr{K}^n$, let $K \mid \xi$ denote the image of the orthogonal projection of the body K onto the vector subspace ξ.

For $K \in \mathscr{K}^n$,

$$W_{n-i}(K) = \frac{\kappa_n}{\kappa_i} \int_{\xi \in \mathrm{Gr}(n, i)} v_i(K \mid \xi) \, d\xi, \tag{1}$$

where κ_i denotes the i-dimensional volume of the unit ball in $\mathbf{R}^i$. The integration in (1) is done with respect to the rotation invariant probability measure on the Grassmannian $\mathrm{Gr}(n, i)$ [19, p. 131].

Two convex bodies K and L are said to be *homothetic* if there exists a positive real number α such that L is a translate of $\alpha K = \{\alpha x : x \in K\}$. If $L = \alpha K$, then K and L are said to be *dilates*. It will be convenient to recall the following inequality for Minkowski mixed volumes. This inequality follows from a successive application of the Aleksandrov–Fenchel Inequality [22, p. 327], followed by Minkowski's Inequality [22, p. 317].

THEOREM 1.4. *Let* $K_1, ..., K_n \in \mathscr{K}^n$. *Then*

$$V(K_1, ..., K_n)^n \geqslant V(K_1) \cdots V(K_n),$$

with equality if and only if $K_1, ..., K_n$ *are homothetic.*

A set function $\mu: \mathscr{K}^n \to \mathbf{R}$ is said to be a *valuation* on $\mathscr{K}^n$ if

1. $\mu(\varnothing) = 0$
2. $\mu(K \cup L) + \mu(K \cap L) = \mu(K) + \mu(L)$

for all $K, L \in \mathscr{K}^n$ such that $K \cup L \in \mathscr{K}^n$ as well.

A valuation μ on $\mathscr{K}^n$ will be called *continuous* if, for $K_i \to K$ in $\mathscr{K}^n$, $\lim_{i \to \infty} \mu(K_i) = \mu(K)$. It is well-known that the elementary mixed volumes $W_0, W_1, ..., W_n$ are continuous valuations on $\mathscr{K}^n$. A valuation μ on $\mathscr{K}^n$ is said to be *rotation invariant* if $\mu(\phi K) = \mu(K)$ for all $\phi \in SO(n)$ and all $K \in \mathscr{K}^n$. Similarly, a valuation μ is said to be *translation invariant* if $\mu(\psi K) = \mu(K)$ for all translations ψ and all $K \in \mathscr{K}^n$. A valuation μ is *invariant under rigid motions* if μ is both translation and rotation invariant.

The following theorem of Hadwiger (see [12, 14, 20]) classifies the continuous valuations on $\mathscr{K}^n$ that are invariant under rigid motions.

THEOREM 1.5. *Suppose that* μ *is a continuous valuation on* $\mathscr{K}^n$, *and that* μ *is invariant under rigid motions. Then there exist* $c_0, c_1, ..., c_n \in \mathbf{R}$ *such that, for all* $K \in \mathscr{K}^n$,

$$\mu(K) = \sum_{i=0}^{n} c_i W_i(K).$$

In other words, the continuous valuations that are invariant under rigid motions form a real vector space spanned by the elementary mixed volumes.

Let $i > 0$. A valuation on $\mathscr{K}^n$ is *homogeneous of degree* i, if $\mu(cA) = c^i \mu(A)$ for all $c \geqslant 0$. In [18], McMullen proved the following theorem.

THEOREM 1.6. *Suppose that* μ *is a continuous translation invariant valuation on* $\mathscr{K}^n$ *that is homogeneous of degree* $n-1$. *Then there exist sequences* $\{L_j\}_{j=0}^{\infty}$ *and* $\{M_j\}_{j=0}^{\infty}$ *in* $\mathscr{K}^n$ *such that*

$$\mu(K) = \lim_{j \to \infty} (V_1(K, L_j) - V_1(K, M_j))$$

for all $K \in \mathscr{K}^n$.

In [10], Goodey and Weil give a similar classification for continuous valuations that are homogeneous of degree 1.

THEOREM 1.7. *Suppose that μ is a continuous translation invariant valuation on $\mathcal{K}^n$. Then μ is homogeneous of degree 1 if and only if there exist sequences $\{L_j\}_{j=0}^{\infty}$ and $\{M_j\}_{j=0}^{\infty}$ in $\mathcal{K}^n$ such that, for all $\delta > 0$,*

$$\mu(K) = \lim_{j\to\infty} (V_1(L_j, K) - V_1(M_j, K))$$

uniformly for all convex bodies $K \subseteq \delta B$.

In the sections that follow we shall develop analogues to Hadwiger's theorem and to the results of McMullen, Goodey, and Weil in the context of star-shaped sets.

2. L^n-STARS

DEFINITION 2.1. A set $A \subseteq \mathbf{R}^n$ is said to be *star-shaped*, if $0 \in A$, and if for each line ℓ passing through the origin in $\mathbf{R}^n$, the set $A \cap \ell$ is a closed interval.

Here 0 denotes the origin in $\mathbf{R}^n$. A star-shaped set A is determined uniquely by its *radial function* $\rho_A \colon \mathbf{S}^{n-1} \to \mathbf{R}$, defined for $u \in \mathbf{S}^{n-1}$ by

$$\rho_A(u) = \max\{\lambda \geqslant 0 : \lambda u \in A\}.$$

If A and C are star-shaped sets, then obviously $A \subseteq C$ if and only if $\rho_A \leqslant \rho_C$.

DEFINITION 2.2. Given star-shaped sets $A_1, ..., A_m$, and positive real numbers $\lambda_1, ..., \lambda_m$, the radial linear combination $A = \lambda_1 A_1 \tilde{+} \lambda_2 A_2 \tilde{+} \cdots \tilde{+} \lambda_m A_m$ is the star-shaped set whose radial function is given by $\rho_A = \lambda_1 \rho_{A_1} + \cdots + \lambda_n \rho_{A_n}$.

Given a sequence of star-shaped sets $A_1, A_2, ...$, and an integer $m > 0$, the sets $\bigcup_{i=1}^m A_i$ and $\bigcap_{i=1}^{\infty} A_i$ are also star-shaped, having radial functions

$$\rho_{A_1 \cup \cdots \cup A_m}(u) = \max_{i \in [m]} \rho_{A_i}(u) \qquad \text{and} \qquad \rho_{A_1 \cap A_2 \cap \cdots}(u) = \inf_{i>0} \rho_{A_i}(u). \tag{2}$$

Note that $\bigcup_{i=1}^{\infty} A_i$ is not necessarily a star-shaped set. If, for all lines ℓ through the origin, the set $\ell \cap \bigcup_{i=1}^{\infty} A_i$ is closed, then the radial function of $\bigcup_{i=1}^{\infty} A_i$ is given by

$$\rho_{A_1 \cup A_2 \cup \cdots}(u) = \max_{i>0} \rho_{A_i}(u).$$

Any non-negative function on $\mathbf{S}^{n-1}$ will determine a star-shaped set, but the set of all non-negative functions is far too large to suit our purposes.

DEFINITION 2.3. Let $p > 0$. A star-shaped set $K \subseteq \mathbf{R}^n$ is an *L^p-star*, if the radial function ρ_K of K is an L^p function on $\mathbf{S}^{n-1}$. Two L^p-stars K, L are defined to be equal whenever $\rho_K = \rho_L$ almost everywhere on $\mathbf{S}^{n-1}$. If ρ_K is a continuous function on $\mathbf{S}^{n-1}$, then K is called a *star body*.

Denote by $\mathscr{S}^n$ the set of all L^n-stars in $\mathbf{R}^n$. Denote by $\mathscr{S}^n_c$ the set of all star bodies in $\mathbf{R}^n$; i.e. the set of all star-shaped sets with continuous radial functions. Both $\mathscr{S}^n$ and $\mathscr{S}^n_c$ are closed under finite unions, finite intersections, and radial combinations. It follows from (2) that the collection $\mathscr{S}^n$ is also closed under countable intersections. A star body is obviously an L^p-star for all $p \geqslant 1$.

DEFINITION 2.4. Let $K_1, K_2, K_3, \ldots \in \mathscr{S}^n$. The sequence $\{K_j\}_1^\infty$ is said to *converge* to the L^n-star K in the *star topology* if $\|\rho_{K_j} - \rho_K\|_n \to 0$ as $j \to \infty$. The convergence of the star sequence $\{K_j\}_1^\infty$ to the L^n-star K shall be denoted $K_j \to K$.

The star topology on $\mathscr{S}^n$ is the natural analogue of the Hausdorff topology on the class $\mathscr{K}^n$ of convex bodies in $\mathbf{R}^n$ defined in Section 1.

Definition 2.4 disagrees with previous definitions of the topology of $\mathscr{S}^n$, in which uniform convergence of radial functions was required for a sequence of star bodies to converge [16]. While sufficient when dealing with star bodies, uniform convergence is too stringent a condition for convergence in the larger class $\mathscr{S}^n$. In particular, note that the action of the rotation group $SO(n)$ on $\mathscr{S}^n$ is *not* continuous in the star topology induced by uniform convergence of radial functions.

DEFINITION 2.5. A set function $\mu: \mathscr{S}^n \to \mathbf{R}$ is a *valuation* if

$$\mu(K \cup L) + \mu(K \cap L) = \mu(K) + \mu(L)$$

for all $K, L \in \mathscr{S}^n$.

Note that a valuation on $\mathscr{S}^n$ need not be countably additive. For $i > 0$, a valuation μ is *homogeneous of degree i*, if $\mu(cA) = c^i \mu(A)$ for all $c \geqslant 0$.

We will denote by V the volume (or Lebesgue measure) in $\mathbf{R}^n$, and we will show that every L^n-star K has a volume $V(K)$. This volume will agree on *convex* bodies K with the volume $V(K)$ mentioned in Section 1. Clearly the volume V is a valuation on $\mathscr{S}^n$. Often it will be convenient to express $V(K)$ in terms of polar coordinates on $\mathbf{R}^n$. Some preliminary definitions are helpful.

DEFINITION 2.6. The *star hull* $st(A)$ of $A \subseteq \mathbf{R}^n$ is defined by

$$st(A) = \{\lambda x : x \in A, 0 \leqslant \lambda \leqslant 1\}.$$

From the definition of star hull we immediately have the following lemma.

LEMMA 2.7. *For all* $A_1, A_2, ... \subseteq \mathbf{R}^n$,

$$st(A_1 \cup A_2 \cup \cdots) = \bigcup_{i=1}^{\infty} st(A_i), \qquad \text{and} \qquad st(A_1 \cap A_2 \cap \cdots) \subseteq \bigcap_{i=1}^{\infty} st(A_i).$$

For $\alpha > 0$, denote by $\alpha\mathbf{S}^{n-1}$ the sphere of radius α, centered at the origin. Similarly, denote by αB the n-dimensional ball of radius α, centered at the origin.

DEFINITION 2.8. Let $\alpha > 0$, and let $A \subseteq \alpha\mathbf{S}^{n-1}$ be measurable with respect to the spherical Lebesgue measure. In this case the star hull $st(A)$ will be called a *spherical cone* with base A and *height* α. A collection of spherical cones $C_1, C_2, ...$ will be called *disjoint* if, $C_i \cap C_j = \{0\}$ for each $i \neq j$.

Note that, by definition, a spherical cone always has a *measurable* base. The results of Lemma 2.7 may be sharpened in the case where the star hulls in question are spherical cones with bases in a common sphere $\alpha\mathbf{S}^{n-1}$.

LEMMA 2.9. *Let* $\alpha > 0$. *For all* $A_1, A_2, ... \subseteq \alpha\mathbf{S}^{n-1}$,

$$st(A_1 \cup A_2 \cup \cdots) = \bigcup_{i=1}^{\infty} st(A_i), \qquad \textit{and} \qquad st(A_1 \cap A_2 \cap \cdots) = \bigcap_{i=1}^{\infty} st(A_i).$$

For $A \subseteq \mathbf{S}^{n-1}$ the *indicator function* $1_A : \mathbf{S}^{n-1} \to \mathbf{R}$ is defined by $1_A(u) = 1$ if $u \in A$, and $1_A(u) = 0$ otherwise.

LEMMA 2.10. *Let* $\alpha > 0$, *and let* $st(A)$ *be the spherical cone with base* $A \subseteq \alpha\mathbf{S}^{n-1}$. *Let* $A_1 = (1/\alpha)\, A = \{x/\alpha : x \in A\}$. *Then* $\rho_{st(A)} = \alpha 1_{A_1}$. *It follows that* $st(A) \in \mathcal{S}^n$.

Note that A_1 is just the radial projection of $st(A) - \{0\}$ onto $\mathbf{S}^{n-1}$.

Let $A \subseteq \mathbf{S}^{n-1}$ be such that $st(A)$ is Lebesgue measurable in $\mathbf{R}^n$. Let $\tilde{S}(A) = nV(st(A))$. It follows from Lemma 2.7, and from the measure properties of V, that $\tilde{S}$ is a countably additive rotation invariant measure on $\mathbf{S}^{n-1}$. These conditions imply that $\tilde{S} = S$ (see [19]). Thus, if $st(A)$ is a Lebesgue measurable subset of $\mathbf{R}^n$, then A is a Lebesgue measurable subset of $\mathbf{S}^{n-1}$, and $st(A)$ is a spherical cone.

DEFINITION 2.11. A polycone P is defined to be a finite union of spherical cones.

It follows from Lemma 2.10 that a polycone is also an L^n-star. Radial functions of polycones are characterized by the following elementary proposition.

PROPOSITION 2.12. *Let P be a polycone. Then there exists a unique collection $\alpha_1, ..., \alpha_m > 0$ and a unique collection of disjoint measurable sets $A_1, ..., A_m \subseteq \mathbf{S}^{n-1}$ such that*

$$\rho_P = \sum_{j=1}^{m} \alpha_j 1_{A_j}.$$

Conversely, any linear combination of measurable indicator functions is the radial function of a polycone.

The set of polycones will prove to be useful for approximating arbitrary L^n-stars.

PROPOSITION 2.13. *Let $K \in \mathscr{S}^n$. Then there exists an increasing sequence $P_1 \subseteq P_2 \subseteq \cdots$ of polycones such that*

$$\lim_{j \to \infty} P_j = K$$

in $\mathscr{S}^n$ and such that $\rho_{P_j} \to \rho_K$ pointwise as well.

When K is a star body, the continuous radial function ρ_K is bounded by some $\alpha > 0$ almost everywhere on $\mathbf{S}^{n-1}$. In this case, an increasing (or decreasing) sequence ρ_j of simple measurable functions may be found that converges to ρ_K *uniformly*. This is no longer true when K is a star-shaped set with an arbitrary L^p radial function.

It is not difficult to show that the polar coordinate formula for the volume of a star body is valid for all L^n-stars:

PROPOSITION 2.14. *For all $K \in \mathscr{S}^n$,*

$$V(K) = \frac{1}{n} \int_{\mathbf{S}^{n-1}} \rho_K^n \, dS.$$

It follows from Proposition 2.14 that volume on the class of L^p-stars is defined if and only if $p = n$.

Analogues to the Minkowski mixed volumes can be defined using radial combinations instead of Minkowski sums. For computing the volume of a radial combination, we have the following dual to Theorem 1.3 (see also [16]).

THEOREM 2.15. *If* $K_1, ..., K_m \in \mathscr{S}^n$, *and if* $\lambda_1, ..., \lambda_m > 0$ *then*

$$V(\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_m K_m) = \sum_{i_1, ..., i_n \in [m]} \tilde{V}(K_{i_1}, ..., K_{i_n}) \, \lambda_{i_1} \cdots \lambda_{i_n},$$

where each coefficient $\tilde{V}(K_{i_1}, ..., K_{i_n})$ *depends only on the bodies* $K_{i_1}, ..., K_{i_n}$.

Given $K_1, ..., K_n \in \mathscr{S}^n$, the coefficient $\tilde{V}(K_1, \ldots K_n)$ given by Theorem 2.15 is called the *dual mixed volume* of $K_1, ..., K_n$. Note that

$$\tilde{V}(K_1, ..., K_n) = \frac{1}{n} \int_{\mathbf{S}^{n-1}} \rho_{K_1} \cdots \rho_{K_n} \, dS. \tag{3}$$

It follows from the integral representation (3) that $\tilde{V}$ is well-defined on $\mathscr{S}^n$, for the dual mixed volumes ignore sets of Lebesgue measure zero.

LEMMA 2.16. *Let* f, g *be non-negative* L^1 *funtions on* $\mathbf{S}^n$. *Let* f_j *be a sequence of non-negative* L^1 *functions such that* $f_j \to f$ *pointwise, and such that*

$$\lim_{j \to \infty} \int_{\mathbf{S}^{n-1}} f_j \, dS = \int_{\mathbf{S}^{n-1}} f \, dS.$$

Let g_j *be a sequence of non-negative* L^1 *functions such that* $g_j \to g$ *pointwise as* $j \to \infty$, *and such that*

$$g_j \leqslant f_j$$

for all j. *Then*

$$\lim_{j \to \infty} \int_{\mathbf{S}^{n-1}} g_j \, dS = \int_{\mathbf{S}^{n-1}} g \, dS.$$

This generalization of the Lebesgue dominated convergence theorem is a simple consequence of Fatou's Lemma [21, p. 23].

PROPOSITION 2.17. *The dual mixed volume* $\tilde{V}(K_1, ..., K_n)$ *is a non-negative continuous function on* $\mathscr{S}^n$ *in* n *variables, and is monotonic with respect to the subset partial ordering on* $\mathscr{S}^n$.

Proof. The non-negativity and monotonicity of $\tilde{V}$ is evident from the integral representation (3). To prove that $\tilde{V}$ is continuous, let $K_1, ..., K_n \in \mathscr{K}^n$, and suppose that for $1 \leqslant i \leqslant n$ the sequence $K_{ij} \to K_i$ as $j \to \infty$. By choosing subsequences if necessary, assume that $\rho_{K_{ij}} \to \rho_{K_i}$ pointwise.

For $\varepsilon > 0$ there exists N such that $\|\rho_{K_{ij}} - \rho_{K_i}\|_n < \varepsilon/n$ for $j \geqslant N$ and $1 \leqslant i \leqslant n$. Let $f_j = \rho_{K_{1j}} + \rho_{K_{2j}} + \cdots + \rho_{K_{nj}}$ for $j > 0$, and let $f = \rho_{K_1} + \rho_{K_2} + \cdots + \rho_{K_n}$. Clearly $f_j \to f$ pointwise, and for $j > N$ we have

$$\|f - f_j\|_n \leqslant \|\rho_{K_1} - \rho_{K_{1j}}\|_n + \cdots + \|\rho_{K_n} - \rho_{K_{nj}}\|_n < \varepsilon.$$

Therefore,

$$\lim_{j \to \infty} \int_{\mathbf{S}^{n-1}} f_j^n \, dS = \int_{\mathbf{S}^{n-1}} f^n \, dS.$$

Since $\rho_{K_1} \cdots \rho_{K_n} \leqslant f^n$ and $\rho_{K_{1j}} \cdots \rho_{K_{nj}} \leqslant f_j^n$ for all j, it follows from Lemma 2.16 that

$$\lim_{j \to \infty} \int_{\mathbf{S}^{n-1}} \rho_{K_{1j}} \cdots \rho_{K_{nj}} \, dS = \int_{\mathbf{S}^{n-1}} \rho_{K_1} \cdots \rho_{K_n} \, dS,$$

so that $\tilde{V}(K_{1j}, ..., K_{nj}) \to \tilde{V}(K_1, ..., K_n)$.

Next, suppose that $\rho_{K_{ij}}$ does *not* converge ρ_{K_i} pointwise. Since every subsequence of the sequence of n-tuples $(K_{1j}, ..., K_{nj})$ contains a sub-sub-sequence whose radial functions converge pointwise [21, p. 68], it follows again that $\tilde{V}K_{1j}, ..., K_{nj}) \to \tilde{V}(K_1, ..., K_n)$. ∎

Two L^p-stars K and L are said to be *dilates* if there exists $c > 0$ such that $\rho_K = c\rho_L$ almost everywhere on $\mathbf{S}^{n-1}$; i.e. if $K = cL$ in $\mathscr{S}^n$. Just as there are radial analogues for the Brunn–Minkowski and Aleksandrov–Fenchel inequalities for star-shaped sets [16], from the Hölder inequality we have the following radial analogue for Theorem 1.4.

THEOREM 2.18. *Let $K_1, K_2, ..., K_n \in \mathscr{S}^n$. Then*

$$\tilde{V}(K_1, K_2, ..., K_n)^n \leqslant V(K_1) V(K_2) \cdots V(K_n),$$

with equality if and only if $K_1, K_2, ..., K_n$ are dilates.

For $0 \leqslant i \leqslant n$ and $K, Q \in \mathscr{S}^n$, define $\tilde{V}_i(K, Q) = \tilde{V}(K, ..., K, Q, ..., Q)$, where K appears $n - i$ times and Q appears i times in the right-hand expression.

PROPOSITION 2.19. *Let $Q \in \mathscr{S}^n$ be fixed. For $0 \leqslant i \leqslant n$, the function $K \mapsto \tilde{V}_i(K, Q)$ is a continuous valuation on $\mathscr{S}^n$.*

We will denote by $\tilde{V}_i(\cdot, Q)$ the valuation on $\mathscr{S}^n$ given by the map $K \mapsto \tilde{V}_i(K, Q)$.

Proof. Since $\rho^k_{K \cup L} + \rho^k_{K \cap L} = \rho^k_K + \rho^k_L$ for all $K, L \in \mathscr{S}^n$ and all $k \geqslant 0$, the integral representation (3) for the dual mixed volumes implies that $\tilde{V}_i(\cdot, Q)$ is a valuation. The continuity of $\tilde{V}_i(\cdot, Q)$ follows from Proposition 2.17. ∎

In analogy to the elementary mixed volumes for convex bodies, the *i-th dual elementary mixed volume* $\widetilde{W}_i(K)$ of an L^n-star K is defined by $\widetilde{W}_i(K)=\widetilde{V}_i(K, B)$, for $0 \leqslant i \leqslant n$. It follows from Proposition 2.19 that the dual elementary mixed volumes are continuous.

In analogy to the mean projection representation (1) for elementary mixed volumes, we have the following mean *intersection* representation [16] for the dual elementary mixed volumes of a star body K:

$$\widetilde{W}_{n-i}(K)=\frac{\kappa_n}{\kappa_i}\int_{\xi \in \mathrm{Gr}(n, i)} v_i(K \cap \xi)\, d\xi.$$

This is one example of the way in which results in the Brunn–Minkowski theory translate into results in the theory of dual mixed volumes. In Section 3 this result is extended to the class of L^n-stars (see Theorem 3.5).

There is a natural action of the special orthogonal group $SO(n)$ on the class of star-shaped sets. The following proposition is an immediate consequence of the definition of a radial function.

Proposition 2.20. *Let $\phi \in SO(n)$, and suppose that K is a star-shaped set. Then the set ϕK is also star-shaped, with $\rho_{\phi K}=\rho_K \circ \phi^{-1}$.*

It follows that ϕK is an L^p-star (or a star body) if and only if K is an L^p-star (or a star body).

3. Haar Measures and the Dual Theory

The dual mixed volumes supply us with a cornucopia of valuations on the lattice of L^n-stars. Section 4 is devoted to the classification of valuations that are homogeneous under dilation of L^n-stars. Such valuations are closely related to measures on the unit sphere $\mathbf{S}^{n-1}$, the subject of the present section. Of especial use is the following well-known theorem [19, p. 138].

Theorem 3.1. *Let μ be a countably additive Borel measure on the unit sphere $\mathbf{S}^{n-1}$, such that μ is invariant under the action of the special orthogonal group $SO(n)$. Then there exists $k \in \mathbf{R}$ such that $\mu=kS$.*

Similarly, there exists a unique countably additive Borel probability measure τ_i on the Grassmannian $\mathrm{Gr}(n, i)$ such that τ_i is rotation invariant.

A *positive integral* μ on $\mathbf{S}^{n-1}$ is a linear functional on the space of continuous real-valued functions on $\mathbf{S}^{n-1}$ such that $\mu(f)$ is positive whenever f is a nonnegative continuous function.

Theorem 3.1 is a special case of a well-known theorem of A. Weil, applied to unit sphere and to the Grassmannian. For a proof of Weil's result, as well as a complete discussion of homogeneous spaces and Haar integrals, see [19, p. 138].

COROLLARY 3.2. *Let $f: \mathbf{S}^{n-1} \to \mathbf{R}$ be an L^1 function. Suppose that, for each $\phi \in SO(n)$, we have $f \circ \phi = f$ almost everywhere. Then there exists $c \in \mathbf{R}$ such that $f = c$ almost everywhere on $\mathbf{S}^{n-1}$.*

An important question may now be settled concerning the conditions one must place on the (fixed) body Q so that the star valuation $\tilde{V}_i(\cdot, Q)$ is rotation invariant.

THEOREM 3.3. *Let $i \in [n]$, and let Q be an L^n-star. The valuation $\tilde{V}_i(\cdot, Q)$ is rotation invariant if and only if Q is a ball; that is, if and only if there exists $c \in \mathbf{R}$ such that $\tilde{V}_i(\cdot, Q) = c\tilde{W}_i$.*

Proof. Let $\phi \in SO(n)$. From Theorem 2.18 it follows that

$$\tilde{V}_i(\phi Q, Q)^n = \tilde{V}(\phi Q, \phi Q, ..., \phi Q, Q, ..., Q) \leqslant V(\phi Q)^{n-i} V(Q)^i = V(Q)^n.$$

Meanwhile, rotation invariance implies that $\tilde{V}_i(\phi Q, Q) = \tilde{V}_i(Q, Q) = V(Q)$. The above inequality becomes an equality. It then follows from the equality conditions of Theorem 2.18 that ϕQ is a dilate of Q. Since ϕ preserves volume, $\phi Q = Q$. Combining this fact with Proposition 2.20, we have $\rho_Q = \rho_{\phi Q} = \rho_Q \circ \phi^{-1}$ almost everywhere on $\mathbf{S}^{n-1}$, for all $\phi \in SO(n)$. By Corollary 3.2, there exists $c \in \mathbf{R}$ such that $\rho^i_Q = c$ almost everywhere. Since ρ_Q is non-negative, $c \geqslant 0$, and so $\rho_Q = c^{1/i}$ almost everywhere. Hence, $Q = c^{1/i} B$. ∎

The following result relates integrable functions on $\mathbf{S}^{n-1}$ to integrable functions on the Grassmannians (see e.g. [13, p. 46]).

LEMMA 3.4. *Let $f: \mathbf{S}^{n-1} \to \mathbf{R}$ be a non-negative L^1 function. Define a function I_f: $\mathrm{Gr}(n, i) \to \mathbf{R}$ by the equation*

$$I_f(\xi) = \int_{\xi \cap \mathbf{S}^{n-1}} f \, dS^{i-1}.$$

The function I_f is defined almost everywhere on $\mathrm{Gr}(n, i)$ with respect to the measure τ_i. Moreover, I_f is integrable with respect to τ_i, with

$$\int_{\mathrm{Gr}(n, i)} I_f \, d\tau_i = \frac{\sigma_{i-1}}{\sigma_{n-1}} \int_{\mathbf{S}^{n-1}} f \, dS.$$

Here σ_j denotes the surface area of the j-dimensional unit sphere $\mathbf{S}^j$.

As an application of Lemma 3.4, we prove the following theorem relating the dual elementary mixed volumes to the *mean intersection integrals* of an L^n-star. This result was originally obtained by Lutwak for star bodies [16].

THEOREM 3.5. *Let $K \in \mathscr{S}^n$. For all $i \in [n]$,*

$$\tilde{W}_{n-i}(K) = \frac{\kappa_n}{\kappa_i} \int_{\xi \in \mathrm{Gr}(n,\, i)} v_i(K \cap \xi)\, d\tau_i.$$

Proof. Let $K \in \mathscr{S}^n$. By Lemma 3.4 there exists an integrable function $I_K = I_{\rho_K^i} : \mathrm{Gr}(n, i) \to \mathbf{R}$, given by the equation

$$I_K(\xi) = \int_{\xi \cap \mathbf{S}^{n-1}} \rho_K^i \, dS^{i-1}. \tag{4}$$

For $K \in \mathscr{S}^n$, it follows from (4) and Proposition 2.14 that

$$\begin{aligned}\tilde{W}_{n-i}(K) &= \frac{\sigma_{n-1}}{n\sigma_{i-1}} \int_{\mathrm{Gr}(n,\, i)} I_K \, d\tau_i = \frac{\kappa_n}{i\kappa_i} \int_{\mathrm{Gr}(n,\, i)} \int_{\xi \cap \mathbf{S}^{n-1}} \rho_K^i \, dS^{i-1}\, d\tau_i \\ &= \frac{\kappa_n}{\kappa_i} \int_{\xi \in \mathrm{Gr}(n,\, i)} v_i(K \cap \xi)\, d\tau_i,\end{aligned}$$

where the first equality follows from Lemma 3.4. ∎

4. HOMOGENEOUS VALUATIONS ON L^n-STARS

We now present a classification theorem for valuations on $\mathscr{S}^n$. Let μ be a continuous valuation on $\mathscr{S}^n$ that is homogeneous of degree i, where $i \in [n]$. Recall that two L^n-stars are equal if their radial functions are equal almost everywhere. If an L^n-star K has Lebesgue measure zero, then $\rho_K = 0$ almost everywhere; i.e. K is equal to the singleton $\{0\}$, so that $\mu(K) = 0$.

Given a homogeneous valuation μ, we construct a *measure* $\tilde{\mu}$ on the sphere $\mathbf{S}^{n-1}$ that is absolutely continuous with respect to the Lebesgue measure on $\mathbf{S}^{n-1}$. The countable additivity of the induced measure $\tilde{\mu}$ will follow from the star continuity of μ. The Lebesgue–Radon–Nikodym Theorem then leads to a classification for the valuation μ.

PROPOSITION 4.1. *Let $\mu : \mathscr{S}^n \to \mathbf{R}$ be a valuation that is continuous and homogeneous of degree k, where $k > 0$. Then μ induces a countably additive* *measure $\tilde{\mu}$ on $\mathbf{S}^{n-1}$ that is absolutely continuous with respect to spherical Lebesgue measure. Moreover, for $A \in \mathcal{S}^n$,*

$$\mu(A) = \int_{\mathbf{S}^{n-1}} \rho_A^k \, d\tilde{\mu}.$$

Proof. Let $A \subseteq \mathbf{S}^{n-1}$ be a Lebesgue measurable set, and let $st(A)$ be the spherical cone with base A and apex 0, as defined in Section 2. Since A is measurable, $st(A)$ is an L^n-star.

Define a measure $\tilde{\mu}$ on all measurable $A \subseteq \mathbf{S}^{n-1}$ by

$$\tilde{\mu}(A) = \mu(st(A)).$$

Recall from Lemma 2.9 that $st(A_1 \cup A_2) = st(A_1) \cup st(A_2)$ and $st(A_1 \cap A_2) = st(A_1) \cap st(A_2)$. If A has Lebesgue measure zero in $\mathbf{S}^{n-1}$, then $st(A) = \{0\}$ in $\mathcal{S}^n$. The homogeneity of μ then implies that $\tilde{\mu}(A) = \mu(st(A)) = 0$. Thus $\tilde{\mu}$ is absolutely continuous with respect to Lebesgue measure.

Finally, let $A_1, A_2, A_3, ...$ be a sequence of measurable subsets of $\mathbf{S}^{n-1}$ that are mutually disjoint. For each i, the function $\rho_{st(A_i)} = 1_{A_i}$. Because the sets A_i are disjoint,

$$1_{A_1 \cup A_2 \cup \cdots} = \sum_{i=1}^{\infty} 1_{A_i} = \lim_{m \to \infty} \sum_{i=1}^{m} 1_{A_i} = \lim_{m \to \infty} 1_{A_1 \cup \cdots \cup A_m},$$

where this limit is taken pointwise on $\mathbf{S}^{n-1}$. In other words,

$$\rho_{st(A_1 \cup A_2 \cup \cdots)} = \lim_{m \to \infty} \rho_{st(A_1 \cup \cdots \cup A_m)},$$

a pointwise limit of radial functions. Since the functions $\rho_{st(A_1 \cup \cdots \cup A_m)}$ form a monotonically increasing sequence, it follows from the monotone convergence theorem that $\|\rho_{st(A_1 \cup A_2 \cup \cdots)} - \rho_{st(A_1 \cup \cdots \cup A_m)}\|_n \to 0$ as $m \to \infty$. Since μ is continuous,

$$\tilde{\mu}\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu\left(\bigcup_{i=1}^{\infty} st(A_i)\right) = \lim_{m \to \infty} \mu\left(\bigcup_{i=1}^{m} st(A_i)\right)$$

$$= \lim_{m \to \infty} \tilde{\mu}\left(\bigcup_{i=1}^{m} A_i\right) = \lim_{m \to \infty} \sum_{i=1}^{m} \tilde{\mu}(A_i) = \sum_{i=1}^{\infty} \tilde{\mu}(A_i),$$

where the first and third equalities follow from Lemma 2.9.

Now let $D \subseteq \alpha\mathbf{S}^{n-1}$, where $\alpha\mathbf{S}^{n-1}$ is the sphere about the origin of radius α. Since μ is k-homogeneous, $\mu(st(D)) = \alpha^k \mu((1/\alpha)\, st(D))$. Since $(1/\alpha)\, st(D)$ is a spherical cone with base $(1/\alpha)\, D \subseteq \mathbf{S}^{n-1}$,

$$\mu(st(D))$$

$$= \alpha^k \mu\left(\frac{1}{\alpha}\, st(D)\right) = \alpha^k \mu\left(st\left(\frac{1}{\alpha}\, D\right)\right) = \alpha^k \tilde{\mu}\left(\frac{1}{\alpha}\, D\right) = \alpha^k \int_{\mathbf{S}^{n-1}} 1_{(1/\alpha)\, D}\, d\tilde{\mu}$$

$$= \alpha^k \int_{\mathbf{S}^{n-1}} \rho^k_{st((1/\alpha)\, D)}\, d\tilde{\mu} = \int_{\mathbf{S}^{n-1}} (\alpha \rho_{st((1/\alpha)\, D)})^k d\tilde{\mu} = \int_{\mathbf{S}^{n-1}} \rho^k_{st(D)}\, d\tilde{\mu}.$$

Let P be a polycone. By Proposition 2.12, $P = \bigcup_{j=1}^m C_j$, a disjoint union of spherical cones C_j. It follows that

$$\mu(P) = \sum_{j=1}^{m} \mu(C_j) = \sum_{j=1}^{m} \int_{\mathbf{S}^{n-1}} \rho^k_{C_j}\, d\tilde{\mu} = \int_{\mathbf{S}^{n-1}} \rho^k_P\, d\tilde{\mu}.$$

Finally, let $K \in \mathscr{S}^n$. By Proposition 2.13, there exists an increasing sequence P_j of polycones such that $P_j \to K$. Since μ is continuous,

$$\mu(K) = \lim_{j \to \infty} \mu(P_j) = \lim_{j \to \infty} \int_{\mathbf{S}^{n-1}} \rho^k_{P_j}\, d\tilde{\mu} = \int_{\mathbf{S}^{n-1}} \rho^k_K\, d\tilde{\mu},$$

where the final equality follows from the monotone convergence theorem. ∎

The situation becomes particularly interesting when the homogeneity degree of μ is an integer $i \in [n]$. Using the Principle of Uniform Boundedness (for bounded linear operators on Hilbert spaces) [21, p. 98] it is not difficult to prove the following:

LEMMA 4.2. *Let $p, q > 1$ such that $(1/p) + (1/q) = 1$, and let $f\colon \mathbf{S}^{n-1} \to \mathbf{R}$. Suppose that*

$$\left| \int_{\mathbf{S}^{n-1}} gf\, dS \right| < \infty,$$

for all $g \in L^p(S)$. Then $f \in L^q(S)$, and thus the linear functional defined by

$$g \mapsto \int_{\mathbf{S}^{n-1}} gf\, dS$$

is a continuous map from $L^p(S)$ into $\mathbf{R}$.

The following theorem relates continuous homogeneous valuations on $\mathscr{S}^n$ to the dual mixed volumes of Section 2.

THEOREM 4.3. *Let $\mu\colon \mathscr{S}^n \to \mathbf{R}$ be a valuation that is continuous and homogeneous of degree i, where $i \in [n]$. Then there exist unique minimal L^n-stars Q_1 and Q_2 such that*

$$\mu(K) = \tilde{V}_{n-i}(K, Q_1) - \tilde{V}_{n-i}(K, Q_2)$$

for all $K \in \mathscr{S}^n$.

Proof. By Proposition 4.1, there exists a measure $\tilde{\mu}$ on $\mathbf{S}^{n-1}$ that is absolutely continuous with respect to S, such that for all $K \in \mathscr{S}^n$,

$$\mu(K) = \int_{\mathbf{S}^{n-1}} \rho_K^i \, d\tilde{\mu}.$$

It then follows from the Lebesgue–Radon–Nikodym Theorem [21, p. 121] that $d\tilde{\mu} = f_\mu \, dS$, where $f_\mu\colon \mathbf{S}^{n-1} \to \mathbf{R}$ is an L^1 function on $\mathbf{S}^{n-1}$. Hence,

$$\mu(K) = \int_{\mathbf{S}^{n-1}} \rho_K^i f_\mu \, dS \tag{5}$$

for all $K \in \mathscr{S}^n$.

Since $|\mu(K)| < \infty$ for all $K \in \mathscr{S}^n$, the integral

$$\left| \int_{\mathbf{S}^{n-1}} g f_\mu \, dS \right| < \infty,$$

for all $L^{n/i}$ functions $g\colon \mathbf{S}^{n-1} \to \mathbf{R}$. It follows from Lemma 4.2 that f_μ is an $L^{n/(n-i)}$ function on $\mathbf{S}^{n-1}$. Conversely, (5) defines a continuous valuation on $\mathscr{S}^n$ for each function $f_\mu \in L^{n/(n-i)}$.

For $u \in \mathbf{S}^{n-1}$, let $f_\mu^+(u) = \max\{f_\mu(u), 0\}$, and let $f_\mu^-(u) = -\min\{f_\mu(u), 0\}$. Since $\|f_\mu^+\|_{n/(n-i)} \leqslant \|f_\mu\|_{n/(n-i)}$ and $\|f_\mu^-\|_{n/(n-i)} \leqslant \|f_\mu\|_{n/(n-i)}$, both f_μ^+ and f_μ^- are $L^{n/(n-i)}$ functions.

Let Q_1 and Q_2 be the star-shaped sets satisfying the conditions $\rho_{Q_1}^{n-i} = n f_\mu^+$ and $\rho_{Q_2}^{n-i} = n f_\mu^-$. Then $Q_1, Q_2 \in \mathscr{S}^n$, and, for all $K \in \mathscr{S}^n$,

$$\begin{aligned}
\mu(K) &= \int_{\mathbf{S}^{n-1}} \rho_K^i f_\mu^+ - \int_{\mathbf{S}^{n-1}} \rho_K^i f_\mu^- \, dS \\
&= \frac{1}{n} \int_{\mathbf{S}^{n-1}} \rho_K^i \rho_{Q_1}^{n-i} \, dS - \frac{1}{n} \int_{\mathbf{S}^{n-1}} \rho_K^i \rho_{Q_2}^{n-i} \, dS \\
&= \tilde{V}_{n-i}(K, Q_1) - \tilde{V}_{n-i}(K, Q_2).
\end{aligned}$$

To see that Q_1 and Q_2 are minimal, suppose there exists another pair of star-shaped sets M_1 and M_2 such that

$$\mu(K) = \tilde{V}_{n-i}(K, M_1) - \tilde{V}_{n-i}(K, M_2) = \frac{1}{n} \int_{\mathbf{S}^{n-1}} \rho_K^i (\rho_{M_1}^{n-i} - \rho_{M_2}^{n-i})\, dS$$

for all $K \in \mathbf{S}^{n-1}$. The uniqueness of f_μ implies that $f_\mu = (1/n)(\rho_{M_1}^{n-i} - \rho_{M_2}^{n-i})$. If $f_\mu(u) \geqslant 0$, then $f_\mu^+(u) = f_\mu(u) \leqslant (1/n)\, \rho_{M_1}^{n-i}(u)$. If $f_\mu(u) < 0$ then $f_\mu^+(u) = 0 \leqslant (1/n)\, \rho_{M_1}^{n-i}(u)$. Hence $(1/n)\, \rho_{Q_1}^{n-i}(u) = f_\mu^+(u) \leqslant (1/n)\, \rho_{M_1}^{n-i}(u)$ for all $u \in \mathbf{S}^{n-1}$ (except possibly on a set of spherical Lebesgue measure zero). It follows that $Q_1 \subseteq M_1$. A similar argument shows that $Q_2 \subseteq M_2$.

Note that the usefulness of Lemma 4.2 to the proof of Theorem 4.3 depends heavily on the fact that the radial functions ρ_K are n-integrable, where n is the dimension of the ambient Euclidean space.

COROLLARY 4.4. *Let $\mu: \mathcal{S}^n \to \mathbf{R}$ be a continuous star valuation, homogeneous of degree i, where $i \in [n]$. If μ is non-negative, then there exists a unique L^n-star Q such that $\mu(K) = \tilde{V}_{n-i}(K, Q)$ for all $K \in \mathcal{S}^n$.*

Proof. In this case the function f_μ in the previous proof will be non-negative almost everywhere (and so we may choose it to be non-negative). Hence $Q_2 = 0$ and $Q = Q_1$. The uniqueness of Q follows from the uniqueness of f_μ in the Lebesgue–Radon–Nikodym Theorem. ∎

There remains the case where the valuation μ is invariant under rotations of L^n-stars. Here the result is particularly satisfying, for it mirrors Hadwiger's classification for valuations on convex bodies (see Theorem 1.5).

THEOREM 4.5. *Let $\mu: \mathcal{S}^n \to \mathbf{R}$ be a continuous rotation invariant valuation that is homogeneous of degree i, where $i \in [n]$. Then there exists $\alpha \in \mathbf{R}$ such that $\mu(K) = \alpha \tilde{W}_{n-i}(K)$, for all $K \in \mathcal{S}^n$.*

Proof. Given $\phi \in SO(n)$, the measure on $\mathbf{S}^{n-1}$ given by $d\tilde{\mu}_\phi = f_\mu \circ \phi\, dS$ is equal to the measure $d\tilde{\mu} = f_\mu\, dS$. By the uniqueness of f_μ in the Lebesgue–Radon–Nikodym Theorem, it follows that $f_\mu \circ \phi = f_\mu$ almost everywhere. By Corollary 3.2, there exists $\alpha \in \mathbf{R}$ such that $f_\mu = \alpha$ almost everywhere. Thus for all $K \in \mathcal{S}^n$,

$$\mu(K) = \frac{1}{n} \int_{\mathbf{S}^{n-1}} \rho_K^i f_\mu\, dS = \frac{\alpha}{n} \int_{\mathbf{S}^{n-1}} \rho_K^i\, dS = \alpha \tilde{W}_{n-i}(K). \quad ∎$$

Examination of the arguments leading up to Theorem 4.5 reveals that this classification applies even if the homogeneity degree i is not an integer, a fact which motivates the following definition.

DEFINITION 4.6. Let $i > 0$, and let Q be a fixed L^n-star. For all $K \in \mathcal{S}^n$, define $\tilde{V}_{n-i}(K, Q)$ by the following expression:

$$\tilde{V}_{n-i}(K, Q) = \frac{1}{n} \int_{\mathbf{S}^{n-1}} \rho_K^i \rho_Q^{n-i} \, dS,$$

and define $\tilde{W}_{n-i}(K) = \tilde{V}_{n-i}(K, B)$.

If $i > n$, note that $\rho_Q > 0$ almost everywhere.

PROPOSITION 4.7. *Let Q be a fixed L^n-star, satisfying the conditions of Definition* 4.6. *Then $\tilde{V}_{n-i}(\cdot, Q)$ defines a continuous valuation on $\mathcal{S}^n$, for $0 \leqslant i \leqslant n$.*

Having at no point used the integer properties of the homogeneity degree i in the proofs of Theorem 4.3 and Theorem 4.5, we may immediately conclude the following.

THEOREM 4.8 (Classification of Homogeneous Valuations on $\mathcal{S}^n$). *Let $\mu : \mathcal{S}^n \to \mathbf{R}$ be a valuation that is continuous and homogeneous of degree i, where $0 \leqslant i \leqslant n$.*

(1) *There exist unique minimal L^n-stars Q_1 and Q_2 such that*

$$\mu(K) = \tilde{V}_{n-i}(K, Q_1) - \tilde{V}_{n-i}(K, Q_2)$$

for all $K \in \mathcal{S}^n$.

(2) *If the valuation μ is also rotation invariant, then there exists $\alpha \in \mathbf{R}$ such that*

$$\mu(K) = \alpha \tilde{W}_{n-i}(K).$$

From Theorem 4.8 we may deduce the following star analogue to the results of McMullen, Goodey, and Weil (Theorems 1.6 and 1.7).

THEOREM 4.9. *Let $\mu : \mathcal{S}^n \to \mathbf{R}$ be a set function. Then μ is a continuous valuation, homogeneous of degree $0 < i < n$, if and only if there exist sequences $\{L_j\}_{j=0}^{\infty}$ and $\{M_j\}_{j=0}^{\infty}$ in $\mathcal{S}_c^n$ such that,*

$$\mu(K) = \lim_{j \to \infty} (\tilde{V}_{n-i}(K, L_j) - \tilde{V}_{n-i}(K, M_j)),$$

for all $K \in \mathcal{S}^n$.

Proof. Let μ be a continuous valuation on $\mathcal{S}^n$ that is homogeneous of degree i, where $0 < i < n$. By Theorem 4.8, there exist fixed L^n-stars Q_1 and Q_2 such that

$$\mu(K) = \tilde{V}_{n-i}(K, Q_1) - \tilde{V}_{n-i}(K, Q_2)$$

for all $K \in \mathscr{S}^n$. Since the set of all real-valued continuous functions on $\mathbf{S}^{n-1}$ is dense in $L^n(S)$, the set $\mathscr{S}^n_c$ of all star bodies is dense in $\mathscr{S}^n$. Therefore, there exist sequences of star bodies $\{L_j\}$ and $\{M_j\}$ such that $L_j \to Q_1$ and $M_j \to Q_2$.

Let $K \in \mathscr{S}^n$. Recall from Proposition 2.19 that $\tilde{V}_{n-i}(K, \cdot): \mathscr{S}^n \to \mathbf{R}$ is continuous. Hence

$$\lim_{j \to \infty} (\tilde{V}_{n-i}(K, L_j) - \tilde{V}_{n-i}(K, M_j)) = (\tilde{V}_{n-i}(K, Q_1) - \tilde{V}_{n-i}(K, Q_2)) = \mu(K).$$

Conversely, suppose that there exist sequences $\{L_j\}_{j=0}^{\infty}$ and $\{M_j\}_{j=0}^{\infty}$ in $\mathscr{S}^n_c$ such that the following limit exists for all $K \in \mathscr{S}^n$:

$$\mu(K) = \lim_{j \to \infty} (\tilde{V}_{n-i}(K, L_j) - \tilde{V}_{n-i}(K, M_j)). \tag{6}$$

For $j > 0$, let $f_j = \rho_{L_j}^{n-i} - \rho_{M_j}^{n-i}$, and let $\mu_j(K) = \tilde{V}_i(K, L_j) - \tilde{V}_{n-i}(K, M_j)$. Note that f_j is continuous. It then follows from the Hölder inequality that each μ_j determines a bounded (continuous) linear operator T_j on the space of $L^{n/i}$ functions on $\mathbf{S}^{n-1}$, given by

$$T_j(g) = \frac{1}{n} \int_{\mathbf{S}^{n-1}} g f_j \, dS,$$

for all $g \in L^{n/i}(\mathbf{S}^{n-1})$. The limit μ also determines a linear operator T on $L^{n/i}(\mathbf{S}^{n-1})$, given by

$$T(g) = \lim_{j \to \infty} T_j(g) = \lim_{j \to \infty} \frac{1}{n} \int_{\mathbf{S}^{n-1}} g f_j \, dS. \tag{7}$$

The existence of the limit $T(g)$ follows from the existence of the limit in (6). Since $|T(g)| < \infty$, the Principle of Uniform Boundedness implies the existence of a constant $c > 0$ such that $|T_j(g)| \leqslant c \|g\|_{n/i}$ for all $g \in L^{n/i}(S)$. It then follows from (7) that $|T(g)| \leqslant c \|g\|_{n/i}$. Therefore, T is a continuous linear functional on $L^{n/i}(S)$, and there exists $f \in L^{n/(n-i)}(S)$ such that

$$\mu(K) = T(\rho_K^i) = \frac{1}{n} \int_{\mathbf{S}^{n-1}} \rho_K^i f \, dS,$$

for all $K \in \mathscr{S}^n$ (see [21, p. 126]). Decompose the function f into the difference of non-negative functions $f = f^+ - f^-$ in the usual way, and let $\rho_{Q_1}^{n-i} = f^+$ and $\rho_{Q_2}^{n-i} = f^-$. Then $Q_1, Q_2 \in \mathscr{S}^n$, and

$$\mu(K) = \frac{1}{n} \int_{\mathbf{S}^{n-1}} \rho_K^i (f^+ - f^-) \, dS = \tilde{V}_{n-i}(K, Q_1) - \tilde{V}_{n-i}(K, Q_2)$$

for all $K \in \mathscr{S}^n$. It now follows from Theorem 4.8 that μ is a continuous valuation, and that μ is homogeneous of degree i. ∎

Acknowledgments

I thank Gian-Carlo Rota for his advice and encouragement in my study of geometric probability. I also thank Richard Dudley, David Jerison, and Daniel Stroock for their help.

References

1. H. Busemann, A theorem on convex bodies of the Brunn–Minkowski type, *Proc. Nat. Acad. Sci. U.S.A.* **35** (1949), 27–31.
2. B. Chen and G.-C. Rota, Totally invariant set functions of polynomial type, *Comm. Pure Appl. Math.* **47** (1994), 187–197.
3. H. Fallert, P. Goodey, and W. Weil, Projections on the sphere and centrally symmetric sets, *Adv. Math.*, in press.
4. R. J. Gardner, Intersection bodies and the Busemann–Petty problem, *Trans. Amer. Math. Soc.* **342** (1994).
5. R. J. Gardner, On the Busemann–Petty problem concerning central sections of centrally symmetric convex bodies, *Bull. Amer. Math. Soc.* **30** (1994).
6. R. J. Gardner, A positive answer to the Busemann–Petty problem in three dimensions, *Ann. Math.* **140** (1994).
7. R. J. Gardner, "Geometric Tomography," Cambridge Univ. Press, Cambridge, 1995.
8. R. J. Gardner and A. Volčič, Tomography of convex and star bodies, *Adv. Math.* **108** (1994), 367–399.
9. P. Goodey, E. Lutwak, and W. Weil, Functional analytic characterizations of classes of convex bodies, *Math. Z.*, to appear.
10. P. Goodey and W. Weil, Distributions and valuations, *Proc. London Math Soc. Ser. 3* **49** (1984).
11. P. Goodey and W. Weil, Intersection bodies and ellipsoids, *Mathematika* **42** (1995), 295–304.
12. H. Hadwiger, "Vorlesungen über Inhalt, Oberfläche, und Isoperimetrie," Springer-Verlag, Berlin, 1957.
13. D. Klain, "Star Measures and Dual Mixed Volumes," Ph. D. thesis, MIT, Cambridge, 1994.
14. D. Klain, A short proof of Hadwiger's characterization theorem, *Mathematika* **42** (1995), 329–339.
15. E. Lutwak, Dual Mixed Volumes, *Pacific J. Math.* **58** (1975), 531–538.
16. E. Lutwak, Intersection bodies and dual mixed volumes, *Adv. Math.* **71** (1988).
17. P. McMullen and R. Schneider, "Valuations on Convex Bodies," *in* "Convexity and Its Applications," (Peter M. Gruber and Jörg M. Wills, Eds.), Birkhäuser Verlag, Boston, 1983.
18. P. McMullen, Continuous translation invariant valuations on the space of compact convex sets, *Arch. Math.* **34** (1980).
19. L. Nachbin, "The Haar Integral," Van Nostrand, New York, 1965.
20. G.-C. Rota, "Introduction to Geometric probability," Lezioni Lincee held at the Scuola Normale Superiore Pisa, Dec. 2–22, 1986.
21. W. Rudin, "Real and Complex Analysis," McGraw–Hill, New York, 1987.
22. R. Schneider, "Convex Bodies: The Brunn–Minkowski Theory," Cambridge Univ. Press., New York, 1993.
23. J. R. Sangwine-Yager, A generalization of outer parallel sets of a convex set, *Proc. Amer. Math. Soc.* **123** (1995), 1559–1564.

24. L. A. SANTALÓ, "Integral Geometry and Geometric Probability," Addison–Wesley, Reading, MA, 1976.
25. R. A. VITALE, $\mathscr{L}^p$ metrics for compact convex sets, *J. Approx. Theory* **45** (1985), 280–287.
26. G. ZHANG, Intersection bodies and the four-dimensional Busemann–Petty problem, *Duke J. Math.* **71** (1993), 233–240.
27. G. ZHANG, Intersection bodies and Busemann–Petty inequalities in $\mathbf{R}^4$, *Ann. Math.* **140**, (1994).
28. G. ZHANG, No polytope is an intersection body, unpublished manuscript.
29. G. ZHANG, Centered bodies and dual mixed volumes, *Trans. Amer. Math. Soc.* **345** (1994), 777–801.