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Invariant Valuations on Star-Shaped Sets

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INTRODUCTION

The Brunn–Minkowski theory of convex bodies and mixed volumes has provided many tools for solving problems involving projections and valuations of compact convex sets in Euclidean space. Among the most beautiful results of twentieth century convexity is Hadwiger's characterization theorem for the elementary mixed volumes (Quermassintegrals); (see [3, 5, 9]). Hadwiger's characterization leads to effortless proofs of numerous results in geometric convexity, including mean projection formulas for convex bodies [13, p. 294] and various kinematic formulas [7, 12, 14, 15]. Hadwiger's theorem also provides a connection between rigid motion invariant set functions and symmetric polynomials [1, 7].

Recently, advancements have been made in a theory introduced by Lutwak [8] that is *dual* to the Brunn–Minkowski theory, a theory tailored for dealing with analogous questions involving star-shaped sets and *intersections* with subspaces (see also [2, 4, 6]). In the dual theory convex bodies are replaced by star-shaped sets, and support functions are replaced by radial functions. Hadwiger's characterizaton theorem is of such fundamental importance that any candidate for a dual theory must possess a dual analogue. However, the dual theory in its original form was not sufficiently rich to be able to accommodate a dual of Hadwiger's theorem.

In [6], it was shown that the natural setting for the dual theory is larger than that envisioned by previous investigators. By defining the dual topology on star-shaped sets in terms of the L^n topology on the space of n-integrable functions on the unit sphere, the author was able to extend the dual theory to a broad class of star-shaped sets, called L^n-stars. Many new theorems can be proved within this larger framework, including a Hadwiger-style classification theorem for continuous valuations on star-shaped sets that are homogeneous with respect to dilation.

In the present paper we discard the stringent requirement of homogeneity and continue with classification theorems for continuous valuations on

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star-shaped sets that are invariant under rotations, as well as those invariant under the action of the special linear group.

As in [6], the background material is sketched, and proofs are given for the main results. For a more detailed treatment, see [4]. The two major results of this paper are presented in Sections 2 and 3.

Section 2 is concerned with the classification of rotation invariant valuations. The collection of all continuous rotation invariant valuations on the L^n-stars turns out to be far larger than the collection of valuations classified by Hadwiger in the convex case. While Hadwiger gave a finite basis for all convex-continuous rigid-motion invariant valuations, the vector space of all star-continuous rotation invariant valuations turns out to have infinite dimension. Not withstanding this breadth of possibility, such valuations remain manageable and even computable. In particular, we show that a continuous rotation invariant star valuation is constructively determined by its behavior when restricted to the set of closed balls with center at the origin.

Section 3 concludes with a classification of all continuous star valuations that are invariant under the action of the group $SL(n)$. This result is especially satisfying: the space of all continuous $SL(n)$-invariant star valuations has only two dimensions, being spanned by the Euler characteristic and the usual volume in $\mathbf{R}^n$.

1. BACKGROUND

We shall denote n-dimensional Euclidean space by $\mathbf{R}^n$. The spherical Lebesgue measure on the $(n-1)$-dimensional unit sphere $\mathbf{S}^{n-1}$ shall be denoted by S. For a function $f\colon \mathbf{S}^{n-1} \to \mathbf{R}$ that is measurable with respect to S, let

$$\|f\|_p = \left(\int_{\mathbf{S}^{n-1}} |f|^p \, dS \right)^{1/p}.$$

A measurable function f on $\mathbf{S}^{n-1}$ is called L^p-integrable, or simply L^p, if $\|f\|_p < \infty$.

Definition 1.1. A set $A \subseteq \mathbf{R}^n$ is said to be *star-shaped*, if A contains the origin, and if for each line l passing through the origin, the set $A \cap l$ is a closed interval.

A star-shaped set A is determined uniquely by its *radial function* $\rho_A\colon \mathbf{S}^{n-1} \to \mathbf{R}$, defined for $u \in \mathbf{S}^{n-1}$ by

$$\rho_A(u) = \max\{\lambda \geqslant 0 \colon \lambda u \in A\}.$$

If A and C are star-shaped sets, then obviously $A \subseteq C$ if and only if $\rho_A \leqslant \rho_C$.

Given a sequence of star-shaped sets $A_1, A_2, ...,$ and an integer $m > 0$, the sets $\bigcup_{i=1}^{m} A_i$ and $\bigcap_{i=1}^{\infty} A_i$ are also star-shaped, having radial functions

$$\rho_{A_1 \cup \cdots \cup A_m}(u) = \max_{1 \leqslant i \leqslant m} \rho_{A_i}(u) \qquad \text{and} \qquad \rho_{A_1 \cap A_2 \cap \cdots}(u) = \inf_{i \geqslant 1} \rho_{A_i}(u). \tag{1}$$

Note that $\bigcup_{i=1}^{\infty} A_i$ is not necessarily a star-shaped set. If for all lines l through the origin, the set $l \cap \bigcup_{i=1}^{\infty} A_i$ is closed and bounded, then the radial function of $\bigcup_{i=1}^{\infty} A_i$ is given by

$$\rho_{A_1 \cup A_2 \cup \cdots}(u) = \max_{i \geqslant 1} \rho_{A_i}(u).$$

Any non-negative function on $\mathbf{S}^{n-1}$ will determine a star-shaped set, but the set of all non-negative functions is far too large to suit our purposes.

Definition 1.2. Let $p > 0$. A star-shaped set $K \subseteq \mathbf{R}^n$ is an L^p-star, if the radial function ρ_K of K is an L^p function on $\mathbf{S}^{n-1}$. Two L^p-stars, K, L are defined to be equal whenever $\rho_K = \rho_L$ almost everywhere on $\mathbf{S}^{n-1}$. If ρ_K is a continuous function on $\mathbf{S}^{n-1}$, then K is called a *star body*.

Denote by $\mathscr{S}^n$ the set of all L^n-stars in $\mathbf{R}^n$. Denote by $\mathscr{S}^n_c$ the set of all star bodies in $\mathbf{R}^n$. Both $\mathscr{S}^n$ and $\mathscr{S}^n_c$ are closed under finite unions and finite intersections. It follows from (1) that the collection $\mathscr{S}^n$ is also closed under countable intersections. A star body is obviously an L^p-star for all $p \geqslant 1$.

Definition 1.3. Let $K_1, K_2, K_3, ..., \in \mathscr{S}^n$. The sequence $\{K_j\}_1^\infty$ is said to *converge* to the L^n-star K in the *star topology*, if $\|\rho_{K_j} - \rho_K\|_n \to 0$ as $j \to \infty$.

Definition 1.4. A set function $\mu \colon \mathscr{S}^n \to \mathbf{R}$ is a *valuation* if

$$\mu(K \cup L) + \mu(K \cap L) = \mu(K) + \mu(L)$$

for all $K, L \in \mathscr{S}^n$.

Note that a valuation need not be countably additive. For $i > 0$, a valuation μ is *homogeneous of degree* i, if $\mu(\alpha A) = \alpha^i \mu(A)$ for all $\alpha \geqslant 0$.

We will use the terms *volume* and *Lebesgue measure* interchangeably in reference to the Lebesgue measure in $\mathbf{R}^n$. Every L^n-star K has a volume, denoted $V(K)$. Clearly the volume V is a valuation on $\mathscr{S}^n$. Often it will be convenient to express $V(K)$ in terms of polar coordinates on $\mathbf{R}^n$. Some preliminary definitions are helpful.

DEFINITION 1.5. The *star hull* $\operatorname{st}(A)$ of $A \subset \mathbf{R}^n$ is defined by

$$\operatorname{st}(A) = \{\lambda x \colon x \in A, 0 \leqslant \lambda \leqslant 1\}.$$

From the definition of star hull we immediately have the following lemma.

LEMMA 1.6. *For all $A_1, A_2, \ldots \subseteq \mathbf{R}^n$,*

$$\operatorname{st}(A_1 \cup A_2 \cup \cdots) = \bigcup_{i=1}^{\infty} \operatorname{st}(A_i) \qquad \text{and} \qquad \operatorname{st}(A_1 \cap A_2 \cap \cdots) \subseteq \bigcap_{i=1}^{\infty} \operatorname{st}(A_i).$$

For $\alpha > 0$, denote by $\alpha\mathbf{S}^{n-1}$ the sphere of radius α, centered at the origin. Similarly, denote by αB the n-dimensional ball of radius α, centered at the origin.

DEFINITION 1.7. Let $\alpha > 0$, and let $A \subseteq \alpha\mathbf{S}^{n-1}$ be measurable with respect to the spherical Lebesgue measure. In this case the star hull $\operatorname{st}(A)$ will be called a *spherical cone* with *base* A and *height* α. A collection of spherical cones $C_1, C_2, \ldots$ will be called *disjoint* if $C_i \cap C_j = \{0\}$ for each $i \neq j$.

Note that, by definition, a spherical cone always has a *measurable* base.

The results of Lemma 1.6 may be sharpened in the case where the star hulls in question are spherical cones with bases in a common sphere $\alpha\mathbf{S}^{n-1}$.

LEMMA 1.8. *Let $\alpha > 0$. For all $A_1, A_2, \ldots \subseteq \alpha\mathbf{S}^{n-1}$,*

$$\operatorname{st}(A_1 \cup A_2 \cup \cdots) = \bigcup_{i=1}^{\infty} \operatorname{st}(A_i) \qquad \text{and} \qquad \operatorname{st}(A_1 \cap A_2 \cap \cdots) = \bigcap_{i=1}^{\infty} \operatorname{st}(A_i).$$

For $A \subseteq \mathbf{S}^{n-1}$, the *indicator function* $1_A \colon \mathbf{S}^{n-1} \to \mathbf{R}$ is defined by $1_A(u) = 1$ if $u \in A$, and $1_A(u) = 0$ otherwise.

LEMMA 1.9. *Let $\alpha > 0$, and let* $\operatorname{st}(A)$ *be the spherical cone with base $A \subseteq \alpha\mathbf{S}^{n-1}$. Let $A_1 = (1/\alpha)\, A = \{x/\alpha \colon x \in A\}$. Then $\rho_{\operatorname{st}(A)} = \alpha 1_{A_1}$. It follows that* $\operatorname{st}(A) \in \mathscr{S}^n$.

Note that A_1 is just the radial projection of $\operatorname{st}(A) - \{0\}$ onto $\mathbf{S}^{n-1}$.

Let $A \subseteq \mathbf{S}^{n-1}$ be such that $\operatorname{st}(A)$ is Lebesgue measurable in $\mathbf{R}^n$. Let $\tilde{S}(A) = nV(\operatorname{st}(A))$. It follows from Lemma 1.6, and from the measure properties of V, that $\tilde{S}$ is a countably additive rotation invariant measure on $\mathbf{S}^{n-1}$. These conditions imply that $\tilde{S} = S$ (see [10]). Thus, if $\operatorname{st}(A)$ is a Lebesgue measurable subset of $\mathbf{R}^n$, then A is a Lebesgue measurable subset of $\mathbf{S}^{n-1}$, and $\operatorname{st}(A)$ is a spherical cone.

DEFINITION 1.10. A *polycone* P is defined to be a finite union of spherical cones.

It follows from Lemma 1.9 that a polycone is also an L^n-star. Radial functions of polycones are characterized by the following elementary proposition.

PROPOSITION 1.11. *Let P be a polycone. Then there exists a unique disjoint collection $\alpha_1, ..., \alpha_m > 0$ and a unique collection of disjoint measurable sets $A_1, ..., A_m \subseteq \mathbf{S}^{n-1}$ such that*

$$\rho_P = \sum_{j=1}^{m} \alpha_j 1_{A_j}.$$

Conversely, any linear combination of measurable indicator functions is the radial function of a polycone.

The set of polycones will prove to be useful for approximating arbitrary L^n-stars.

PROPOSITION 1.12. *Let $K \in \mathscr{S}^n$. Then there exists an increasing sequence $P_1 \subseteq P_2 \subseteq \ldots$ of polycones such that*

$$\lim_{j \to \infty} P_j = K$$

in $\mathscr{S}^n$ and such that $\rho_{P_j} \to \rho_K$ pointwise as well.

Proof. Since ρ_K is an L^n function on $\mathbf{S}^{n-1}$, there exists an increasing sequence of non-negative simple measurable functions ρ_j on $\mathbf{S}^{n-1}$ such that $\lim_{j \to \infty} \rho_j = \rho_K$, a pointwise limit of functions. By Proposition 1.11, each ρ_j is the radial function of a polycone P_j. Since the ρ_j are increasing, $P_j \to K$ in $\mathscr{S}^n$, and $P_i \subseteq P_j$ whenever $i < j$. ∎

It is not difficult to show that the polar coordinate formula for the volume of a star body is valid for all L^n-stars:

PROPOSITION 1.13. *For all $K \in \mathscr{S}^n$,*

$$V(K) = \frac{1}{n} \int_{\mathbf{S}^{n-1}} \rho_K^n \, dS.$$

It follows from Proposition 1.13 that volume on the class of L^p-stars is defined if and only if $p \geqslant n$.

There is a natural action of the special linear group $SL(n)$ on the class of star-shaped sets. This action is especially nice when restricted to the special orthogonal group $SO(n)$. We begin with some preliminary results. (For detailed arguments, see [4, p. 32], [11]).

Proposition 1.14. *Let $f\colon \mathbf{S}^{n-1} \to \mathbf{R}$ be an L^p function, where $p \geqslant 1$, and let $\zeta\colon \mathbf{S}^{n-1} \to \mathbf{S}^{n-1}$ be a diffeomorphism. Then the composed function $f \circ \zeta\colon \mathbf{S}^{n-1} \to \mathbf{R}$ is an L^p function.*

Proof. Suppose that $p=1$. Define a set function v on the Borel subsets of $\mathbf{S}^{n-1}$ as follows. For all $A \subseteq \mathbf{S}^{n-1}$, define

$$v(A) = S(\zeta^{-1}(A)).$$

Since ζ^{-1} is a diffeomorphism, ζ^{-1} maps open sets to open sets and closed sets to closed sets. Moreover, ζ^{-1} commutes with unions and intersections, for ζ^{-1} is a bijective function on $\mathbf{S}^{n-1}$. It follows that ζ^{-1} maps Borel sets to Borel sets, and that v is a Borel measure on $\mathbf{S}^{n-1}$. If $S(A)=0$, then $S(\zeta^{-1}(A))=0$ as well ([4, p. 32], [11]), so that $v(A)=0$. In other words, v is a Borel measure that is absolutely continuous with respect to the invariant measure S on $\mathbf{S}^{n-1}$. By the Lebesgue–Radon–Nikodym theorem [11, p. 121], there exists an L^1 function $g_v\colon \mathbf{S}^{n-1} \to \mathbf{R}$, such that

$$v(A) = \int_{\mathbf{S}^{n-1}} 1_A g_v \, dS$$

for all Borel sets $A \subseteq \mathbf{S}^{n-1}$. Since v is a non-negative measure, $g_v \geqslant 0$.

Meanwhile, suppose that $h\colon \mathbf{S}^{n-1} \to \mathbf{R}$ is a *continuous* function. In this case,

$$\int_{\mathbf{S}^{n-1}} h g_v \, dS = \int_{\mathbf{S}^{n-1}} h \, dv = \int_{\mathbf{S}^{n-1}} h \circ \zeta \, dS = \int_{\mathbf{S}^{n-1}} h J_\zeta \, dS,$$

where J_ζ is the Jacobian of ζ. In other words, $g_v = J_\zeta$. But J_ζ is a continuous function on $\mathbf{S}^{n-1}$. In particular, J_ζ is *bounded* on $\mathbf{S}^{n-1}$. Hence, there exists $M>0$ such that $0 \leqslant g_v \leqslant M$.

Since $f\colon \mathbf{S}^{n-1} \to \mathbf{R}$ is an L^1 function,

$$\int_{\mathbf{S}^{n-1}} f \circ \zeta \, dS = \int_{\mathbf{S}^{n-1}} f \, dv = \int_{\mathbf{S}^{n-1}} f g_v \, dS \leqslant M \int_{\mathbf{S}^{n-1}} f \, dS < \infty.$$

In other words, $f \circ \zeta$ is an L^1 function.

Next suppose that f is an L^p function, where $p>1$. Then f^p is an L^1 function. It follows that $f^p \circ \zeta = (f \circ \zeta)^p$ is an L^1 function, so that $f \circ \zeta$ is L^p. ∎

PROPOSITION 1.15. *Let $\phi \in SL(n)$. For all star-shaped sets K, the set ϕK is also star-shaped. Moreover, for all $u \in \mathbf{S}^{n-1}$,*

$$\rho_{\phi K}(u) = \frac{1}{|\phi^{-1}(u)|} \rho_K\left(\frac{\phi^{-1}(u)}{|\phi^{-1}(u)|}\right).$$

It follows that ϕK is an L^p-star (or a star body) if and only if K is an L^p-star (or a star body). If $\phi \in SO(n)$, then $\rho_{\phi K} = \rho_K \circ \phi^{-1}$.

Proof. Suppose that K is a star-shaped set. Since ϕ is linear and bijective, $\phi(0) = 0$, and for all lines l through the origin in $\mathbf{R}^n$, ϕ maps the closed line segment $K \cap l$ to the closed line segment $\phi K \cap \phi l$. It follows that ϕK is star-shaped.

For all $u \in \mathbf{S}^{n-1}$,

$$\rho_{\phi K}(u) = \frac{1}{|\phi^{-1}(u)|} \rho_K\left(\frac{\phi^{-1}(u)}{|\phi^{-1}(u)|}\right).$$

It follows that $\rho_{\phi K}$ is a continuous function if and only if ρ_K is continuous.

Suppose that $K \in \mathcal{S}^p$. Let $\zeta : \mathbf{S}^{n-1} \to \mathbf{S}^{n-1}$ be given by

$$\zeta(u) = \frac{\phi^{-1}(u)}{|\phi^{-1}(u)|}.$$

Since ζ is a diffeomorphism on $\mathbf{S}^{n-1}$, we may apply Proposition 1.14 to conclude that $\rho_K \circ \zeta$ is L^p. The function $1/|\phi^{-1}(u)|$ is continuous on $\mathbf{S}^{n-1}$ and is therefore bounded. It follows that the function

$$\rho_{\phi K}(u) = \frac{1}{|\phi^{-1}(u)|} \rho_K\left(\frac{\phi^{-1}(u)}{|\phi^{-1}(u)|}\right) \tag{2}$$

is an L^p function, and that ϕK is an L^p-star.

If $\phi \in SO(n)$, then ϕ preserves length, so that $|\phi^{-1}(u)| = |u| = 1$. It follows from (2) that $\rho_{\phi K}(u) = \rho_K(\phi^{-1}(u))$. ∎

For additional background material on star-shaped sets and the dual Brunn–Minkowski theory, see [2, 4, 6, 8]

2. ROTATION INVARIANT VALUATIONS ON L^n-STARS

We now present a classification theorem for valuations on $\mathcal{S}^n$ that are rotation invariant.

DEFINITION 2.1. Let $K \in \mathcal{S}^n$. The L^n-star K is bounded if there exists $\alpha > 0$ such that $\rho_K < \alpha$ almost everywhere on $\mathbf{S}^{n-1}$. Denote by $\mathcal{S}^n_b$ the set of all bounded L^n-stars.

The following lemma will be useful for the classification of the rotation invariant valuations.

LEMMA 2.2. *Let $g: \mathbf{R} \to \mathbf{R}$ be a continuous function. The inequality*

$$\left| \int_{\mathbf{S}^{n-1}} g \circ \rho \, dS \right| < \infty \tag{3}$$

holds for all non-negative L^n functions $\rho: \mathbf{S}^{n-1} \to \mathbf{R}$, if and only if there exist $a, b \geqslant 0$ such that $|g(x)| \leqslant ax^n + b$ for all $x \geqslant 0$.

Proof. To begin, suppose that (3) holds for all $\rho \in L^n(\mathbf{S}^{n-1})$, and suppose also that there do *not* exist $a, b \geqslant 0$ such that $|g(x)| \leqslant ax^n + b$ for all $x \geqslant 0$. Then for all integers $k > 0$ there exists $\alpha_k \geqslant 1$ such that $|g(\alpha_k)| > k\alpha_k^n$. Without loss of generality, we may assume that for all $k > 0$ there exists $\alpha_k \geqslant 1$ such that $g(\alpha_k) > k\alpha_k^n$ (if not, then replace the function g with $-g$ and proceed).

Since $k > 0$, this statement is equivalent to the following claim:

For all $k > 0$ there exists $\alpha_k \geqslant 1$ such that $g(\alpha_k) > 2^k \alpha_k^n$.

Let $U_1, U_2, \ldots$ be a sequence of disjoint open subsets of $\mathbf{S}^{n-1}$ such that $S(U_k) = 1/2^k \alpha_k^n$, for all $k > 0$. Let $Z = \mathbf{S}^{n-1} - \bigcup_{k>0} U_k$.

Define a function $\rho: \mathbf{S}^{n-1} \to \mathbf{R}$ as follows. For all $u \in \mathbf{S}^{n-1}$, set $\rho(u) = \alpha_k$ if $u \in U_k$. If $u \in Z$ then set $\rho(u) = 0$. If then follows that

$$\int_{\mathbf{S}^{n-1}} \rho^n \, dS = \sum_{k>0} \int_{U_k} \rho^n \, dS = \sum_{k>0} \alpha_k^n S(U_k) = \sum_{k>0} \alpha_k^n \frac{1}{2^k \alpha_k^n} = \sum_{k>0} \frac{1}{2^k} = 1.$$

In other words, the function ρ is a non-negative L^n function on $\mathbf{S}^{n-1}$. Meanwhile,

$$\begin{aligned} \int_{\mathbf{S}^{n-1}} g \circ \rho \, dS &= \int_Z g \circ \rho \, dS + \sum_{k>0} \int_{U_k} g \circ \rho \, dS \\ &= g(0)\, S(Z) + \sum_{k>0} g(\alpha_k)\, S(U_k) \\ &> g(0)\, S(Z) + \sum_{k>0} 2^k \alpha_k^n \frac{1}{2^k \alpha_k^n} = \infty. \end{aligned}$$

In other words, $g \circ \rho$ does not satisfy (3), contradicting our assumption. Therefore, there must exist $a, b \geqslant 0$ such that $|g(x)| \leqslant ax^n + b$ for all $x > 0$.

The converse is trivial. ∎

PROPOSITION 2.3. *Suppose that μ is a continuous rotation invariant valuation on $\mathcal{S}^n$, such that $\mu(\{0\}) = 0$. Then there exists a unique continuous function $g\colon [0, \infty) \to \mathbf{R}$ such that for all $K \in \mathcal{S}^n$,*

$$\mu(K) = \int_{\mathbf{S}^{n-1}} g \circ \rho_K \, dS. \tag{4}$$

Moreover, there exist $a, b > 0$ such that $|g(x)| \leqslant ax^n + b$ for all $x \geqslant 0$.

Proof. The continuous rotation invariant valuation μ induces a countably additive measure $\tilde{\mu}$ on the unit sphere $\mathbf{S}^{n-1}$ defined as follows. For $A \subseteq \mathbf{S}^{n-1}$, define

$$\tilde{\mu}(A) = \mu(\mathrm{st}(A)).$$

Since $\mu(\{0\}) = 0$, it is evident that $\tilde{\mu}$ is absolutely continuous with respect to spherical Lebesgue measure [4, p. 54], [6]. Since $\tilde{\mu}$ is also rotation invariant, there exists $g_1 \in \mathbf{R}$ such that $\tilde{\mu} = g_1 S$, where S denotes the spherical Lebesgue measure. This is a consequence of the uniqueness of Haar measure on homogeneous spaces (see [10]).

This construction may be applied to each sphere centered at the origin. For all $\alpha > 0$, denote by S_α the Lebesgue measure on $\alpha\mathbf{S}^{n-1}$. The valuation μ induces an invariant measure $\tilde{\mu}_\alpha$ on $\alpha\mathbf{S}^{n-1}$, defined by

$$\tilde{\mu}_\alpha(A) = \mu(\mathrm{st}(A))$$

for all measurable $A \subseteq \alpha\mathbf{S}^{n-1}$. Once again, $\tilde{\mu}_\alpha = g_\alpha S_\alpha$ on $\alpha\mathbf{S}^{n-1}$, where g_α is a real constant.

In other words, given a spherical cone C with base $A \subseteq \alpha\mathbf{S}^{n-1}$ and apex at the origin,

$$\mu(C) = \tilde{\mu}_\alpha(A) = g_\alpha S_\alpha(A) = g_\alpha \int_{\mathbf{S}^{n-1}} \rho_C^{n-1} \, dS = \int_{\mathbf{S}^{n-1}} g_{\rho_C} \rho_C^{n-1} \, ds.$$

Let P be a polycone. By Proposition 1.11, there exist disjoint spherical cones $C_1, ..., C_m$ such that $P = \bigcup_{j=1}^m C_j$, and $\rho_P = \sum_{i=1}^m \rho_{C_i}$. From the argument above it follows that

$$\mu(P) = \sum_{i=1}^m \mu(C_i) = \sum_{i=1}^m \int_{\mathbf{S}^{n-1}} g_{\rho_{C_i}} \rho_{C_i}^{n-1} \, dS = \int_{\mathbf{S}^{n-1}} g_{\rho_P} \rho_P^{n-1} \, dS,$$

where the last equality follows from the fact that the cones C_i are disjoint.

Let $g\colon [0, \infty) \to \mathbf{R}$ be defined by $g(x) = g_x x^{n-1}$. The expression above now becomes

$$\mu(P) = \int_{\mathbf{S}^{n-1}} g \circ \rho_P \, dS,$$

for any polycone P.

The function g is determined uniquely by the action of the valuation μ on balls centered at the origin. This follows from the fact that

$$\mu(\alpha B) = \int_{\mathbf{S}^{n-1}} g \circ \rho_{\alpha B} \, dS = \int_{\mathbf{S}^{n-1}} g(\alpha) \, dS = g(\alpha) \, \sigma_{n-1},$$

where αB is the ball of radius α, and where σ_{n-1} is the surface area of the sphere $\mathbf{S}^{n-1}$. Since the valuation μ is continuous, the expression $\mu(\alpha B)$ defines a function on the positive reals that is continuous in the variable α. It follows that g is a continuous function. Since $\mu(\{0\}) = 0$, it follows that $g(0) = 0$.

Next, let $K \in \mathscr{S}^n_b$. By Proposition 1.12, there exists an increasing sequence of polycones P_i such that $P_i \to K$ and such that $\rho_{P_i} \to \rho_K$ pointwise as $i \to \infty$. Since (4) holds for each P_i, the continuity of g and the Lebesgue dominated convergence theorem then imply that (4) holds for K as well.

Finally, let $K \in \mathscr{S}^n$. For all $j \geqslant 0$, let $E_j = \{u \in \mathbf{S}^{n-1} : 0 \leqslant \rho_K(u) \leqslant j\}$, and let K_j be the bounded L^n-star with radial function $\rho_j = 1_{E_j} \rho_K$. Since the sets E_j form an increasing sequence with respect to inclusion, the bounded functions ρ_j also form an increasing sequence, such that

$$\lim_{j \to \infty} \rho_j = \lim_{j \to \infty} 1_{E_j} \rho_K = \rho_K \lim_{j \to \infty} 1_{E_j} = \rho_K.$$

Therefore $K_j \to K$ in $\mathscr{S}^n$, and $\mu(K_j) \to \mu(K)$.

Since $g(0) = 0$, we have

$$g(\rho_j(u)) = g(1_{E_j}(u) \, \rho_K(u)) = g(\rho_K(u)),$$

if $u \in E_j$, and

$$g(\rho_j(u)) = g(1_{E_j}(u) \, \rho_K(u)) = g(0) = 0,$$

if $u \notin E_j$. In other words,

$$g \circ \rho_j = g(1_{E_j} \rho_K) = (1_{E_j})(g \circ \rho_K) = (1_{[0, j]} \circ \rho_K)(g \circ \rho_K). \tag{5}$$

From (5) it is clear that $g \circ \rho_j(u) \to g \circ \rho_K(u)$ *monotonically* at each $u \in \mathbf{S}^{n-1}$. Since each K_j is a *bounded* subset of $\mathbf{R}^n$, the previous argument implies that

$$\mu(K_j) = \int_{\mathbf{S}^{n-1}} g \circ \rho_j \, dS,$$

for all $j > 0$. The monotone convergence theorem and the continuity of μ then imply that

$$\mu(K) = \lim_{j \to \infty} \mu(K_j) = \lim_{j \to \infty} \int_{\mathbf{S}^{n-1}} g \circ \rho_j \, dS = \int_{\mathbf{S}^{n-1}} g \circ \rho_K \, dS.$$

Since $\mu(K)$ takes on finite values for all $K \in \mathscr{S}^n$, we have

$$|\mu(K)| = \left| \int_{\mathbf{S}^{n-1}} g \circ \rho_K \, dS \right| < \infty,$$

for all non-negative L^n functions $\rho_K \colon \mathbf{S}^{n-1} \to \mathbf{R}$. It then follows from Lemma 2.2 that there exist $a, b > 0$ such that $|g(x)| \leqslant ax^n + b$ for all $x \geqslant 0$. ∎

So far this classification is one-sided. To each continuous rotation invariant valuation μ on $\mathscr{S}^n$ (such that $\mu(\{0\}) = 0$) we have associated a unique continuous function $g \colon [0, \infty) \to \mathbf{R}$ (such that $g(0) = 0$), satisfying the inequality conditions of Proposition 2.3. This injective mapping from the rotation invariant valuations to the "sub nth degree" continuous functions on the nonnegative reals is in fact a *bijective* mapping. In order to see this, we will require the following lemma.

LEMMA 2.4. *Let f, g be non-negative L^1 functions on $\mathbf{S}^{n-1}$. Let f_i be a sequence of non-negative L^1 functions such that $f_i \to f$ pointwise, and such that*

$$\lim_{i \to \infty} \int_{\mathbf{S}^{n-1}} f_i \, dS = \int_{\mathbf{S}^{n-1}} f \, dS.$$

Let g_i be a sequence of non-negative L^1 functions such that $g_i \to g$ pointwise as $i \to \infty$ and such that

$$g_i \leqslant f_i$$

for all i. Then

$$\lim_{i \to \infty} \int_{\mathbf{S}^{n-1}} g_i \, dS = \int_{\mathbf{S}^{n-1}} g \, dS.$$

This generalization of the Lebesgue dominated convergence theorem is a simple consequence of Fatou's Lemma.

PROPOSITION 2.5. *Suppose that $g\colon [0, \infty) \to \mathbf{R}$ is a continuous function such that $g(0)=0$, and suppose that there exist $a, b>0$ such that $|g(x)| \leqslant ax^n + b$ for all $x \geqslant 0$. Let μ be defined by the equation*

$$\mu(K) = \int_{\mathbf{S}^{n-1}} g \circ \rho_K \, dS,$$

for all $K \in \mathscr{S}^n$. Then μ is a continuous rotation invariant valuation on $\mathscr{S}^n$. Moreover, $\mu(\{0\}) = 0$.

Proof. Let $K \in \mathscr{S}^n$. Since ρ_K is a non-negative L^n function on $\mathbf{S}^{n-1}$, it follows from Lemma 2.2 that

$$\left| \int_{\mathbf{S}^{n-1}} g \circ \rho_K \, dS \right| < \infty,$$

so that $|\mu(K)| < \infty$.

Obviously, for $K, L \in \mathscr{S}^n$, and $u \in \mathbf{S}^{n-1}$,

$$g \circ \rho_{K \cup L}(u) + g \circ \rho_{K \cap L}(u) = g \circ \rho_K(u) + g \circ \rho_L(u),$$

and hence, μ is a valuation on $\mathscr{S}^n$.

To show that μ is rotation invariant, let $K \in \mathscr{S}^n$, and let $\phi \in SO(n)$. Proposition 1.15 and the rotation invariance of the Haar measure S imply that

$$\mu(\phi K) = \int_{\mathbf{S}^{n-1}} g \circ \rho_{\phi K} \, dS = \int_{\mathbf{S}^{n-1}} g \circ \rho_K \circ \phi^{-1} \, dS = \int_{\mathbf{S}^{n-1}} g \circ \rho_K \, dS = \mu(K).$$

In other words, μ is a rotation invariant valuation on $\mathscr{S}^n$. It remains to show that μ is continuous.

Assume first that $g \geqslant 0$. Let $K_m, K \in \mathscr{S}^n$ such that $K_m \to K$ as $m \to \infty$. Suppose that $\{K_i\}$ is a *subsequence* of $\{K_m\}$. Evidently there exists a *sub*-subsequence $\{K_{i_j}\}$, with radial functions denoted ρ_{i_j}, such that $\rho_{i_j} \to \rho_K$ pointwise.

For all $j>0$ let $g_j = g \circ \rho_{i_j}$, and let $f_j = a\rho_{i_j}^n + b$. Similarly, let $g_K = g \circ \rho_K$, and let $f_K = a\rho_K^n + b$. Then,

$$\mu(K_{i_j}) = \int_{\mathbf{S}^{n-1}} g \circ \rho_{i_j} \, dS = \int_{\mathbf{S}^{n-1}} g_j \, dS,$$

and

$$\mu(K)=\int_{\mathbf{S}^{n-1}} g\circ\rho_K\,dS=\int_{\mathbf{S}^{n-1}} g_K\,dS.$$

Since g is a continuous function, $g_j\to g_K$ pointwise. Similarly, $f_j\to f_K$ pointwise. Note also that $0\leqslant g_j\leqslant f_j$, and $0\leqslant g_K\leqslant f_K$. Since $K_{i_j}\to K$, Lemma 2.4 applies, so that

$$\lim_{j\to\infty}\int_{\mathbf{S}^{n-1}} g_j\,dS=\int_{\mathbf{S}^{n-1}} g_K\,dS.$$

In other words, for every subsequence $\{K_i\}$ of the original sequence $\{K_m\}$, there exists a sub-subsequence $\{K_{i_j}\}$ such that

$$\lim_{j\to\infty}\mu(K_{i_j})=\lim_{j\to\infty}\int_{\mathbf{S}^{n-1}} g_j\,dS=\int_{\mathbf{S}^{n-1}} g_K\,dS=\mu(K).$$

It then follows that

$$\lim_{m\to\infty}\mu(K_m)=\mu(K),$$

so that μ is continuous.

Suppose finally that g is an arbitrary continuous function on $\mathbf{R}$ such that $g(0)=0$ and such that $|g(x)|\leqslant ax^n+b$ for all $x\geqslant 0$. Let $g^+(x)=\max\{g(x),0\}$ and $g^-(x)=\max\{-g(x),0\}$, for all $x\geqslant 0$. Then g can be expressed as the difference $g=g^+-g^-$, where g^+ and g^- are non-negative continuous functions. Note that $g^+(0)=g^-(0)=g(0)=0$. This decomposition of g induces a decomposition $\mu=\mu^+-\mu^-$, where each of μ^+ and μ^- are continuous valuations by the previous argument. Therefore, μ is also continuous. ∎

In other words, the injective correspondence given by Proposition 2.3 is, in fact, a bijection. These results are summarized in the following theorem.

THEOREM 2.6. *Let μ be a continuous valuation on $\mathscr{S}^n$ that is invariant under rotations. Let $\mu(\{0\})=0$. Then there exists a unique continuous function $g\colon[0,\infty)\to\mathbf{R}$ such that for all $K\in\mathscr{S}^n$,*

$$\mu(K)=\int_{\mathbf{S}^{n-1}} g\circ\rho_K\,dS. \tag{6}$$

The function g satisfies the following two conditions:

- $g(0)=0$.
- *There exist $a, b>0$ such that $|g(x)|\leqslant ax^n+b$ for all $x\geqslant 0$.*

Conversely, (6) *defines a continuous rotation invariant valuation* μ, *for all continuous functions* $g\colon [0, \infty) \to \mathbf{R}$ *satisfying the above conditions.*

In order for the preceding arguments to flow gracefully, it was necessary to assume that $\mu(\{0\}) = 0$. We now examine the case $\mu(\{0\}) \neq 0$.

DEFINITION 2.7. Define the valuation $\chi\colon \mathscr{S}^n \to \mathbf{R}$, as follows. For each L^n-star K, define $\chi(K) = 1$.

This constant set function is obviously continuous and rotation invariant. That χ is a valuation is also clear.

Now let μ be any continuous rotation invariant valuation on $\mathscr{S}^n$. Let $c = \mu(\{0\})$. Since the valuation $v = \mu - c\chi$ satisfies the conditions of Theorem 2.6, there exists a unique continuous function $g\colon [0, \infty) \to \mathbf{R}$ such that $g(0) = 0$, and such that

$$\mu(K) = v(K) + c = \int_{\mathbf{S}^{n-1}} g \circ \rho_K \, dS + c = \int_{\mathbf{S}^{n-1}} \left(g \circ \rho_K + \frac{c}{\sigma_{n-1}} \right) dS$$

for all $K \in \mathscr{S}^n$.

Hence, there is a unique continuous function $G = g + (c/\sigma_{n-1})$ such that

$$\mu(K) = \int_{\mathbf{S}^{n-1}} G \circ \rho_K \, dS$$

for all $K \in \mathscr{S}^n$.

Note once again that the function G is determined uniquely by the action of the valuation μ on balls centered at the origin. As in the proof of Proposition 2.3,

$$\mu(\alpha B) = \int_{\mathbf{S}^{n-1}} G \circ \rho_{\alpha B} \, dS = \int_{\mathbf{S}^{n-1}} G(\alpha) \, dS = G(\alpha) \, \sigma_{n-1}, \tag{7}$$

where αB is the ball of radius α, and where σ_{n-1} is the surface area of the sphere $\mathbf{S}^{n-1}$.

Since the functions G and g differ only by a constant, the condition that $|g|$ is bounded above by a polynomial of the form $ax^n + b$ (where $a, b \geqslant 0$) is equivalent to the same condition on the function $|G|$.

This result is summarized in the following theorem.

THEOREM 2.8 (Classification of Rotation Invariant Valuations). *There is a bijective correspondence between continuous valuations* μ *on* $\mathscr{S}^n$ *that are invariant under rotations and continuous functions* $G\colon [0, \infty) \to \mathbf{R}$ *such that* $|G(x)| \leqslant ax^n + b$ *for some* $a, b \geqslant 0$.

This correspondence is given by the following equations:

$$\mu(K) = \int_{\mathbf{S}^{n-1}} G \circ \rho_K \, dS$$

for all $K \in \mathcal{S}^n$, *and*

$$G(\alpha) = \frac{1}{\sigma_{n-1}} \mu(\alpha B)$$

for all $\alpha \geqslant 0$.

Many important properties of μ translate into analogous properties of the associated function G. The following corollary is an immediate consequence of (7).

COROLLARY 2.9. *Let μ be a continuous valuation on $\mathcal{S}^n$ that is invariant under rotations. Let G: $[0, \infty) \to \mathbf{R}$ be the continuous function associated to μ in Theorem* 2.8.

- *The valuation μ is non-negative if and only if the function G is non-negative on* $[0, \infty)$.
- *The valuation μ is monotonic on $\mathcal{S}^n$ if and only if G is an increasing function on* $[0, \infty)$.
- *The valuation μ is positively homogeneous of degree $0 \leqslant \alpha \leqslant n$ if and only if there exists $c \in \mathbf{R}$ such that $G(x) = cx^\alpha$.*

Note that if μ is homogeneous of degree $0 \leqslant i \leqslant n$, where i is an *integer*, then for all $K \in \mathcal{S}^n$,

$$\mu(K) = cn\widetilde{W}_{n-i}(K),$$

where $\widetilde{W}_{n-i}(K)$ denotes the $(n-i)$th dual elementary mixed volume of the L^n-star K (see [4, p. 30], [6, 8]).

Let $\mathrm{Gr}(n, i)$ denote the Grassmannian of i-dimensional subspaces of $\mathbf{R}^n$, and let v_i denote the i-dimensional volume in the subspace ξ. In this case, it is known that

$$\mu(K) = \frac{cn\kappa_n}{\kappa_i} \int_{\xi \in \mathrm{Gr}(n, i)} v_i(K \cap \xi) \, d\xi.$$

See also [4, p. 66; 6; 8]. Here κ_i denotes the i-dimensional volume of the unit ball in $\mathbf{R}^i$.

3. $SL(n)$-INVARIANT VALUATIONS ON L^n-STARS

Recall from Proposition 1.15 that the special linear group $SL(n)$ acts on $\mathscr{S}^n$. This fact motivates an investigation of $SL(n)$-invariant valuations.

To begin, note that such a valuation μ is rotation invariant. Theorem 2.8 then implies the existence of an associated continuous function $G\colon [0, \infty) \to \mathbf{R}$ such that for all $K \in \mathscr{S}^n$,

$$\mu(K) = \int_{\mathbf{S}^{n-1}} G \circ \rho_K \, dS.$$

LEMMA 3.1. *Let μ be an $SL(n)$-invariant continuous valuation on $\mathscr{S}^n$. Suppose that $\mu(\{0\}) = 0$. Then μ is either a non-negative valuation or a non-positive valuation.*

Proof. Let G be the continuous function associated to μ, as discussed above. If $\mu = 0$, then we are done. If not, assume without loss of generality that $G(a) > 0$, for some $a > 0$. Suppose there exists $y \in (0, a)$ such that $G(y) < 0$. Since G is continuous, G attains a minimum $G(r_0) = m < 0$ on $[0, a]$.

Let B_0 be the ball of radius r_0, centered at the origin. Let $\lambda = a/r_0$. Let $T \in SL(n)$ be the map represented by a diagonal matrix, with n diagonal entries $\lambda, 1/\lambda, 1, ..., 1$. Finally, let $E = T(B_0)$. In other words, E is the image of the ball B_0 under the linear map T, an ellipsoid. Since $0 < r_0 < a$, we know that $1/\lambda < 1 < \lambda$.

Let ∂E denote the boundary of E. For all $y \in \partial E$, we have $y = Tx$ for some $x \in \partial B_0 = r_0 \mathbf{S}^{n-1}$. Hence,

$$|y| = |Tx| = \left(\lambda^2 x_1^2 + \frac{1}{\lambda^2} x_2^2 + x_3^2 + \cdots + x_n^2 \right)^{1/2} \leqslant \lambda \, |x| = \lambda r_0 = a.$$

It follows that the image $\operatorname{Im}(\rho_E) \subseteq [0, a]$. For $x = (r_0, 0, ..., 0)$, we have $|Tx| = |(a, 0, ..., 0)| = a$, so that $a = \rho_E(x/|x|) \in \operatorname{Im}(\rho_E)$.

Since G is minimized at r_0 on $[0, a]$,

$$\mu(E) = \int_{\mathbf{S}^{n-1}} G \circ \rho_E \, dS \geqslant \int_{\mathbf{S}^{n-1}} G(r_0) \, dS = m\sigma_{n-1}.$$

Meanwhile, the $SL(n)$-invariance of μ implies that $\mu(E) = \mu(B_0) = m\sigma_{n-1}$. The inequality is an equality. This can only be so if $G(x) = m$ almost everywhere on the closed interval $\operatorname{Im}(\rho_E)$. Since G is continuous, it follows that $G(x) = m$ on $\operatorname{Im}(\rho_E)$. In particular, $G(a) = m < 0$. This is a contradiction.

Thus G must be non-negative on $[0, a]$. Now suppose that $G(b) < 0$ for some $b > a$. The same argument, applied to the valuation $-\mu$ on the interval

$[0, b]$, implies that this is also impossible. It follows that G is a non-negative function. By Corollary 2.9, μ is a non-negative valuation. ▮

The trick of deforming a ball of local minimum or maximum value into a convenient ellipse will be used again in the proof of this next lemma.

LEMMA 3.2. *Let μ be an $SL(n)$-invariant continuous valuation on $\mathcal{S}^n$. Suppose that $\mu(\{0\})=0$. Then μ is monotonic.*

Proof. If $\mu=0$, we are done. Suppose that $\mu \neq 0$. By Lemma 3.1, one may assume without loss of generality that μ is a non-negative valuation. Let G be the non-negative continuous function associated to μ. Let $a>0$. Since G is continuous, G attains a maximum $G(r_1)=M$ on $[0, a]$.

Let B_1 be the ball of radius r_1, centered at the origin. Let $\lambda=a/r_1$. Let $T \in SL(n)$ be the map represented by a diagonal matrix, with n diagonal entries λ, $1/\lambda$, 1, ..., 1. Finally, let $E=T(B_1)$. In other words, E is the image of the ball B_1 under the linear map T, an ellipsoid. In this case $0<r_1 \leqslant a$, so that $1/\lambda \leqslant 1 \leqslant \lambda$.

We follow an argument almost identical to that given in the proof of Lemma 3.1.

As before, for all $y \in \partial E$, we have $y=Tx$ for some $x \in \partial B_1 = r_1 \mathbf{S}^{n-1}$. Hence,

$$|y|=|Tx|=\left(\lambda^2 x_1^2+\frac{1}{\lambda^2} x_2^2+x_3^2+\cdots+x_n^2\right)^{1/2} \leqslant \lambda\,|x|=\lambda r_1=a.$$

It follows that the image $\mathrm{Im}(\rho_E) \subseteq [0, a]$. For $x=(r_1, 0, ..., 0)$, we have $|Tx|=|(a, 0, ..., 0)|=a$, so that $a=\rho_E(x/|x|) \in \mathrm{Im}(\rho_E)$.

Since G is maximized at r_1 on $[0, a]$,

$$\mu(E)=\int_{\mathbf{S}^{n-1}} G \circ \rho_E \, dS \leqslant \int_{\mathbf{S}^{n-1}} G(r_1)\, dS = M\sigma_{n-1}.$$

Meanwhile, the $SL(n)$-invariance of μ implies that $\mu(E)=\mu(B_1)=M\sigma_{n-1}$. The inequality is an equality. This can only be so if $G(x)=M$ almost everywhere on the closed interval $\mathrm{Im}(\rho_E)$. Since G is continuous, we have $G(x)=M$ on $\mathrm{Im}(\rho_E)$. In particular, $G(a)=M$.

Thus, on any closed interval $[0, a]$, where $a>0$, the function G attains its maximum at a. It follows that if $0 \leqslant a \leqslant b$, then $G(a) \leqslant G(b)$. In other words, G is a monotonic function. By Corollary 2.9, μ is a monotonic valuation. ▮

THEOREM 3.3. *Let μ be an $SL(n)$-invariant continuous valuation on $\mathcal{S}^n$. Suppose that $\mu(\{0\})=c_0$. Then $\mu=c_0\chi+c_1 V$, where $c_0, c_1 \in \mathbf{R}$ are constants.*

Here V denotes volume in $\mathbf{R}^n$, and χ denotes the constant unit valuation on L^n-stars, defined in the previous section.

Proof. Suppose that $\mu(\{0\})=0$. Let G be the continuous function associated to μ. Lemma 3.2 and Corollary 2.9 imply that G is a monotonic function on $[0, \infty)$. Let $\alpha=\mu(B)$, where B is the unit ball, centered at the origin. For all $K \in \mathcal{S}^n$, define

$$\nu(K)=\mu(K)-\frac{\alpha}{V(B)} V(K).$$

Since the valuation ν satisfies the conditions of Lemma 3.2, ν must be a monotonic valuation. But $\nu(\{0\})=0$, and $\nu(B)=0$. Hence $\nu(K)=0$ for all L^n-stars $K \subseteq B$. In other words, $\mu(K)=(\alpha/V(B))\, V(K)$ for all L^n-stars $K \subseteq B$.

This argument may be repeated, using larger and larger balls centered at the origin, instead of the unit ball B. Since each larger ball will contain the last, the constant $\alpha/V(B)$ must never change. Since every polycone is contained in some ball centered at the origin,

$$\mu(P)=\frac{\alpha}{V(B)} V(P)$$

for all polycones P. It then follows from Proposition 1.11 and from the continuity of μ and V that

$$\mu(K)=\frac{\alpha}{V(B)} V(K)$$

for all $K \in \mathcal{S}^n$. Let $c_0=0$ and $c_1=\alpha/V(B)$.

Next suppose that $\mu(\{0\})=c_0 \neq 0$. Repeat the preceding argument, using the valuation $\mu-c_0\chi$. It follows that there exists $c_1 \in \mathbf{R}$ such that $\mu-c_0\chi=c_1 V$. ∎

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REFERENCES

1. B. Chen and G.-C. Rota, Totally invariant set functions of polynomial type, *Comm. Pure Appl. Math.* **47** (1994), 187–197.

2. R. J. Gardner, “Geometric Tomography,” Cambridge Univ. Press, New York, 1995.
3. H. Hadwiger, “Vorlesungen über Inhalt, Oberfläche, und Isoperimetrie,” Springer-Verlag, Berlin, 1957.
4. D. Klain, “Star Measures and Dual Mixed Volumes,” Ph.D. thesis, MIT, Cambridge, 1994.
5. D. Klain, A short proof of Hadwiger’s characterization theorem, *Mathematika* **42** (1995), 329–339.
6. D. Klain, Star valuations and dual mixed volumes, *Adv. Math.* **121** (1996), 80–108.
7. D. Klain and G.-C. Rota, “Introduction to Geometric Probability,” Cambridge Univ. Press, New York, in press.
8. E. Lutwak, Intersection bodies and dual mixed volumes, *Adv. Math.* **71**, No. 2 (1988), 232–261.
9. P. McMullen and R. Schneider, Valuations on convex bodies, *in* “Convexity and Its Applications” (P. M. Gruber and J. M. Wills, Eds.), pp. 170–247, Birkhäuser, Boston, 1983.
10. L. Nachbin, “The Haar Integral,” Van Nostrand, New York, 1965.
11. W. Rudin, “Real and Complex Analysis,” McGraw–Hill, New York, 1987.
12. L. A. Santaló, “Integral Geometry and Geometric Probability,” Addison–Wesley, Reading, MA, 1976.
13. R. Schneider, “Convex Bodies: The Brunn–Minkowski Theory,” Cambridge Univ. Press, New York, 1993.
14. R. Schneider and J. A. Wieacker, Integral geometry, *in* “Handbook of Convex Geometry” (P. M. Gruber and J. M. Wills, Eds.), pp. 1349–1390, North-Holland, Amsterdam, 1993.
15. W. Weil, Kinematic integral formulas for convex bodies, *in* “Contributions to Geometry, Proc. Geometry Symposium Siegen, 1978” (J. Tölke and J. M. Wills, Eds.), pp. 60–76, Birkhäuser, Boston, 1978.