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NOTES

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A Probabilistic Proof of the Spherical Excess Formula

Daniel A. Klain

Abstract. A probabilistic proof of Girard's angle excess formula for the area of a spherical triangle emerges from the observation that an unbounded 3-dimensional convex cone, with single vertex at the origin, has only three kinds of 2-dimensional orthogonal projections: a 2-dimensional convex cone with one vertex, a 2-dimensional half-plane (an outcome with probability zero), and a 2-dimensional plane.

A triangle T in the unit sphere with inner angles θ_1, θ_2, and θ_3 has area given by the *spherical excess formula*[1]:

$$\mathtt{Area}(T) = \theta_1 + \theta_2 + \theta_3 - \pi. \tag{1}$$

See Figure 1.

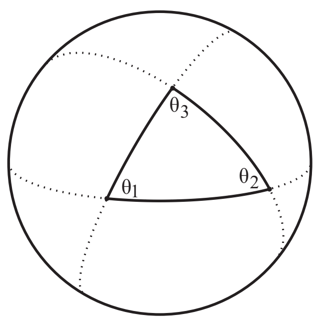

Figure 1. A spherical triangle.

This note offers a probabilistic proof of the angle excess formula (1), based on the observation that an unbounded cone at the origin in $\mathbb{R}^3$ has only three kinds of 2-dimensional orthogonal projections: a cone in $\mathbb{R}^2$, a half-plane in $\mathbb{R}^2$ (an outcome with probability zero), and all of $\mathbb{R}^2$. See Figure 2.

Observe that, if we omit the middle outcome of measure zero, the number of edges on each projected figure is *twice* the number of vertices.

Some notation will help to interpret angles as probabilities. Let $\mathbb{S}$ denote the unit sphere in $\mathbb{R}^3$ centered at the origin, having surface area 4π.

[1]This formula was discovered in 1603 by Thomas Harriot [**6**, p. 65] and is also known as Girard's formula [**2**, p. 95].

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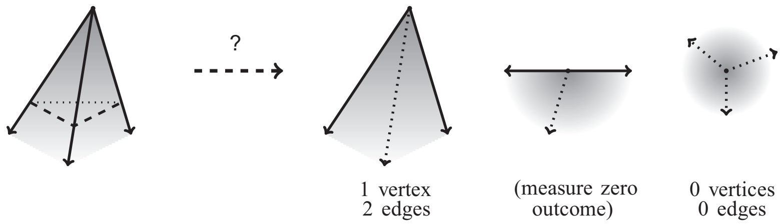

Figure 2. Projections of a 3-dimensional cone.

Suppose that P is a convex polytope in $\mathbb{R}^3$, and let x be any point of P. The *solid inner angle* $a_P(x)$ of P at x is given by

$$a_P(x) = \{u \in \mathbb{S} \mid \ x + \epsilon u \in P \quad \text{for some } \epsilon > 0\}.$$

Let $\alpha_P(x)$ denote the measure of the solid inner angle $a_P(x) \subseteq \mathbb{S}$, given by the usual surface area measure on subsets of the sphere.

If F is a vertex, edge, or facet of a convex polytope P, then the solid inner angle measure $\alpha_P(x)$ is the same at every point x in the relative interior of F. This value will be denoted by $\alpha_P(F)$.

Consider the case of an unbounded cone C with single vertex at the origin o, as in Figure 2. Specifically, let v_1, v_2, v_3 be three linearly independent unit vectors in $\mathbb{R}^3$, and let C denote all nonnegative linear combinations:

$$C = \{t_1 v_1 + t_2 v_2 + t_3 v_3 \mid t_i \geq 0\}.$$

The polyhedral cone C has exactly one vertex at o and three (unbounded) edges e_i along the directions of the vectors v_i. Note that $\alpha_C(o)$ is the area of the spherical triangle with vertices at v_i. Denote the spherical angles of this triangle by θ_i, as in Figure 1 (where o lies at the center of the sphere in Figure 1).

Given a uniformly distributed random unit vector u, let C_u denote the orthogonal projection of C onto the plane $u^\perp$. Evidently C_u will resemble one of the outcomes in Figure 2. Specifically, C_u will cover the entire plane $u^\perp$ if and only if u lies in the interior of $\pm a_C(o)$. It follows that $C_u = u^\perp$ with probability

$$\frac{\texttt{Area}(a_C(o)) + \texttt{Area}(-a_C(o))}{4\pi} = \frac{2\alpha_C(o)}{4\pi} = \frac{\alpha_C(o)}{2\pi}.$$

Since the number of vertices of C_u is either 0 or 1, the expected number of vertices of C_u is given by the complementary probability

$$E(\#\text{ of vertices}) \ = \ 1 - \frac{\alpha_C(o)}{2\pi}. \tag{2}$$

Meanwhile, an edge e projects to the interior of C_u if and only if u lies in the interior of $\pm a_C(e)$. Taking the complement as before, e projects to a boundary edge of C_u with probability $1 - \frac{\alpha_C(e)}{2\pi}$. Observe that each solid inner angle measure $\alpha_C(e_i)$ is given by $2\theta_i$ (see Figure 3), so that the expected number of edges of C_u is

$$E(\#\text{ of edges}) \ = \ \sum_i \left(1 - \frac{\alpha_C(e_i)}{2\pi}\right) \ = \ \sum_i \left(1 - \frac{\theta_i}{\pi}\right) \ = \ 3 - \frac{\theta_1 + \theta_2 + \theta_3}{\pi}. \tag{3}$$

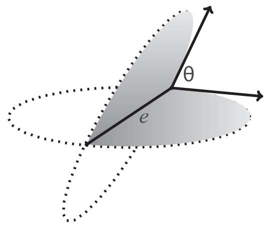

Figure 3. $\alpha_C(e) = 2\theta$.

Since the number of edges in C_u is almost surely *twice* the number of vertices (see Figure 2), the identities (2) and (3) imply that

$$3 - \frac{\theta_1 + \theta_2 + \theta_3}{\pi} \;=\; E(\#\text{ of edges}) \;=\; 2E(\#\text{ of vertices}) \;=\; 2 - \frac{\alpha_C(o)}{\pi}. \tag{4}$$

It is now immediate from (4) that

$$\alpha_C(o) \;=\; \theta_1 + \theta_2 + \theta_3 - \pi,$$

as asserted in (1).

In higher dimensions a proliferation of cases makes this viewpoint much more complicated. However, variations of this approach are applied in [**1**], [**3**], [**4**, p. 315a], [**5**], and [**7**] to derive many fundamental formulas for intrinsic volumes of polyhedral angles and cones.

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