# TETRAHEDRA WITH CONGRUENT FACE PAIRS

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Abstract. If the four triangular facets of a tetrahedron can be partitioned into pairs having the same area, then the triangles in each pair must be congruent to one another. A Heron-style formula is then derived for the volume of a tetrahedron having this kind of symmetry.

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From elementary geometry we learn that two triangles are congruent if their edges have the same three lengths. In particular, there is only one congruence class of *equilateral* triangles having a given edge length. Said differently, any pair of equilateral triangles in the Euclidean plane are *similar*, differing at most by an isometry and a dilation. Meanwhile, triangles that are symmetric under a single reflection have two congruent sides and are said to be *isosceles*.

The situation is more complicated in higher dimensions. Indeed, an analogous characterization of 3-dimensional tetrahedra already leads to 25 different symmetry classes [22]. These tetrahedral symmetry classes are of special interest in organic chemistry [8, 9, 21], and conditions for tetrahedral symmetry based on the measures of dihedral angles have also been explored [23].

A tetrahedron in $\mathbb{R}^3$ is *equilateral* or *regular* if all of its edges have the same length. More generally, a tetrahedron is said to be *isosceles* if all four triangular facets are congruent to one another, or, equivalently, if opposing (non-incident) edges have the same length. Isosceles tetrahedra are also known as *disphenoids* [4, p. 15]. It has been shown that if all four facets of a tetrahedron $T$ have the same *area*, then $T$ must be isosceles [10, p. 94][11, 16].

Consider the following more general symmetry class of tetrahedra: A tetrahedron $T$ will be called *reversible* if its facets are congruent in pairs; that is, if the facets of $T$ can be labelled $f_1, f_2, f_3, f_4$, where $f_1 \cong f_2$ and $f_3 \cong f_4$.

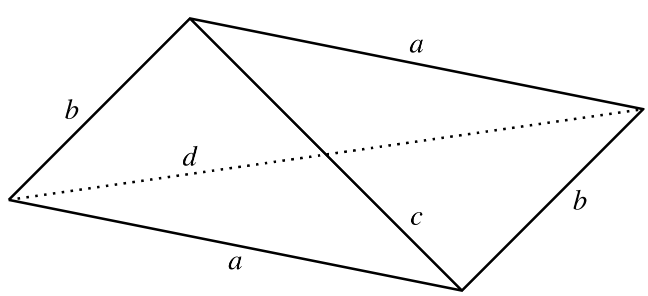


Figure 1. An reversible tetrahedron with edge lengths $a, a, b, b, c, d$.

In this note we show that, as in the isosceles case, reversible tetrahedra are characterized by the areas of their facets: if the four triangular facets of $T$ can be partitioned into pairs with the same area, then those pairs consist of congruent facets.

In the final section we give an intuitive method for deriving a Heron-style factorization of the volume of a reversible tetrahedron in terms of its edge lengths.

## 1. Facets normals and areas determine tetrahedra

The following proposition will allow us to exploit symmetries more easily.

**Proposition 1.1.** *Suppose that a tetrahedron $T$ has outward facet unit normals $u_0, u_1, u_2, u_3$, with corresponding facet areas $\alpha_0, \alpha_1, \alpha_2, \alpha_3 > 0$. Then*

$$\alpha_0 u_0 + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = 0. \tag{1}$$

*Conversely, if unit vectors $u_0, u_1, u_2, u_3$ span $\mathbb{R}^3$, and if $\alpha_i > 0$ satisfy (1), then there exists a tetrahedron $T$, having outward facet unit normals $u_i$, and corresponding facet areas $\alpha_i$, and this tetrahedron is unique up to translation.*

This proposition is a very special case of the Minkowski Existence Theorem, which plays a central role in the Brunn-Minkowski theory of convex bodies, and is somewhat difficult to prove [3, 18]. However, this special case for tetrahedra is a simple consequence of linear algebra.

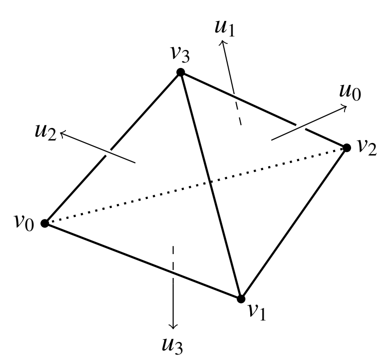


FIGURE 2. A tetrahedron with outward unit normals $u_i$.

*Proof.* Let $T$ be a tetrahedron with vertices at $v_0, v_1, v_2, v_3 \in \mathbb{R}^3$, where $v_0 = o$, the origin. Let us assume the vertices are labelled so that $v_1, v_2, v_3$ have a positive ("right-handed") orientation.

Denote by $u_0, u_1, u_2, u_3$ the outward unit normal vectors of the facets of $T$, where $u_i$ is associated with the facet opposite to the vertex $v_i$, as in Figure 2. Let $\alpha_i$ denote the area of that same

$i$th facet. Since $v_0 = o$, we have

$$\begin{aligned} v_2 \times v_3 &= -2\alpha_1 u_1 \\ v_3 \times v_1 &= -2\alpha_2 u_2 \\ v_1 \times v_2 &= -2\alpha_3 u_3 \\ (v_3 - v_1) \times (v_2 - v_1) &= -2\alpha_0 u_0. \end{aligned} \tag{2}$$

After summing both sides of these equations the identity (1) now follows.

To prove the converse, suppose we are given unit vectors $u_0, u_1, u_2, u_3$ that span $\mathbb{R}^3$ and $\alpha_i > 0$ satisfying (1). Let $\tilde{T}$ denote the intersection of the closed half-spaces $x \cdot u_i \leq 1$.

The spanning condition on the $u_i$, along with the identity (1), imply that any 3 of the vectors $u_i$ are linearly independent. Since each $\alpha_i > 0$, it follows from (1) that $\tilde{T}$ is a bounded tetrahedron with facets normal to the $u_i$. Translate this tetrahedron so that one vertex lies at the origin $o$, and then slide the facet opposite to $o$ along the direction of $u_0$ so that this facet has area $\alpha_0$. This new tetrahedron $T$ now has facet areas $\alpha'_0 = \alpha_0, \alpha'_1, \alpha'_2$, and $\alpha'_3$, and must also satisfy (1), so that

$$\alpha_0 u_0 + \alpha'_1 u_1 + \alpha'_2 u_2 + \alpha'_3 u_3 = 0.$$

Combining this with identity (1) for the original given data yields

$$\alpha'_1 u_1 + \alpha'_2 u_2 + \alpha'_3 u_3 = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3.$$

Since $u_1, u_2, u_3$ are linearly independent, each $\alpha'_i = \alpha_i$, and $T$ is the tetrahedron required, unique up to translation. □

**Remark:** Given the surface data $u_i$ and $\alpha_i$ it is not difficult to construct the corresponding tetrahedron $T$ explicitly. To do so, let $C$ denote the $3 \times 3$ matrix having columns $-2\alpha_i u_i$ for $i > 0$, ordered so that $C$ has positive determinant. The matrix

$$A = \det(C)^{\frac{1}{2}} C^{-t}$$

has cofactor matrix $C$. It is not difficult to show (using Cramer's Rule and basic linear algebra) that the columns of $A$, along with the origin, yield the vertices of a tetrahedron having facet normals $u_i$ and corresponding facet areas $\alpha_i$. Uniqueness up to translation also follows from this explicit construction (which generalizes to $n$ dimensions as well).

## 2. Equal areas imply congruent faces

We now prove that the areas of the facets alone will determine if a tetrahedron is reversible.

**Theorem 2.1.** *Suppose that $T$ is a tetrahedron in $\mathbb{R}^3$, and denote by $f_1, f_2, f_3, f_4$ the triangular facets of $T$. If the facets of $T$ satisfy the conditions*

$$Area(f_1) = Area(f_2) \quad and \quad Area(f_3) = Area(f_4)$$

*then $f_1 \cong f_2$ and $f_3 \cong f_4$.*

The proof of Theorem 2.1 uses the method given by McMullen in [16] to verify the special case in which all four facets have the same area (as in Corollary 2.2 below).

*Proof.* Denote by $u_i$ the outward unit normal vector to the facet $f_i$ of $T$. Suppose that $Area(f_1) = Area(f_2) = \alpha$ and $Area(f_3) = Area(f_4) = \beta$, where $\alpha, \beta > 0$. The identity (1) asserts that

$$\alpha u_1 + \alpha u_2 + \beta u_3 + \beta u_4 = 0.$$

Denote

$$w = \alpha u_1 + \alpha u_2 = -\beta u_3 - \beta u_4.$$

Let $\psi$ denote the rotation of $\mathbb{R}^3$ by the angle $\pi$ around the the axis through $w$. Since the vectors $\alpha u_1$ and $\alpha u_2$ have the same length, the points $o, \alpha u_1, \alpha u_2, w$ are the vertices of a rhombus. The rotation $\psi$ rotates this rhombus onto itself, exchanging the vectors $\alpha u_1$ and $\alpha u_2$. The points $o, \beta u_3, \beta u_4, -w$ form a rhombus through the same axis, so that $\psi$ also exchanges the vectors $\beta u_3$ and $\beta u_4$. Since $\psi$ is a rotation, it preserves orthogonality. It follows that $P$ and $\psi P$ have the same normal vectors and the same corresponding facet areas. Proposition 1.1 then implies that $P$ and $\psi P$ are congruent by a translation. In particular, the facets $f_1$ and $f_2$ are congruent, as are $f_3$ and $f_4$. □

The case of isosceles tetrahedra described in the introduction follows as an immediate corollary to Theorem 2.1.

**Corollary 2.2.** *Suppose that $T$ is a tetrahedron in $\mathbb{R}^3$. If the faces $f_i$ of $T$ satisfy the condition*

$$Area(f_1) \;=\; Area(f_2) \;=\; Area(f_3) \;=\; Area(f_4)$$

*then $f_1 \cong f_2 \cong f_3 \cong f_4$.*

In other words, if a tetrahedron $T$ is equiareal, then $T$ is also isosceles. For alternative proofs and variants of Corollary 2.2, see [10, 11, 15, 16].

**Remark:** Corollary 2.2 has long been known to have an analogue in which area is replaced by *perimeter*. The proof is very simple: If all of the facets of $T$ have the same perimeter, the resulting system of linear equations (in the six edge lengths of $T$) implies that opposing edges must have the same length, so that $T$ is isosceles. A similar argument shows that if the facets of $T$ can be partitioned into pairs having the same perimeter then $T$ is reversible.

## 3. **Factoring the volume**

Suppose that $T \subseteq \mathbb{R}^3$ is a tetrahedron with vertices at $v_0, v_1, v_2, v_3 \in \mathbb{R}^3$, where $v_0 = o$, the origin. As before, let $A$ denote the matrix whose columns are given by the vectors $v_i$, and suppose that the $v_i$ are ordered so that $A$ has positive determinant. The volume of $T$ is then given by $\det(A) = 6V(T)$, so that

$$V(T)^2 = \frac{1}{36}\det(A^tA).$$

The entries of the matrix $A^tA$ are dot products of the form $v_i \cdot v_j$. From the identity,

$$2v_i \cdot v_j \;=\; |v_i|^2 + |v_j|^2 - |v_i - v_j|^2 \tag{3}$$

it then follows that the value of $V(T)^2$ is a *polynomial* in the *squares* of the edge lengths of $T$. Said differently, if $T$ has edge lengths $a_{ij}$ (the distance between vertices $v_i$ and $v_j$), then $V(T)^2$ is a polynomial in the variables $b_{ij} = a_{ij}^2$, as well as the variables $a_{ij}$ themselves. This polynomial is sometimes formulated in terms of linear algebraic expressions such as Cayley-Menger determinants [19, p. 125]. While the Cayley-Menger heuristic outlined above applies in arbitrary dimension, the 3-dimensional case has been known at least as far back as Piero della Francesca [17].[1]

In certain instances, the polynomial $V(T)^2$ admits factorization into linear or quadratic irreducible factors. For the 2-dimensional case, the area $A(\Delta)$ of a triangle $\Delta$ having edge lengths $a, b, c$ is given by

$$A(\Delta)^2 = \frac{1}{16}(a+b+c)(-a+b+c)(a-b+c)(a+b-c),$$

a factorization known as *Heron's formula* [5, p. 58]. Although the 3-dimensional case is more complicated [7], there exist non-trivial factorizations of $V(T)^2$ when the tetrahedron $T$ satisfies the symmetry properties examined in the previous section.

For example, if $T$ is an isosceles tetrahedron, having edge lengths $a, b, c$ (each repeated twice in pairs of opposing edges), then

$$V(T)^2 = \frac{1}{72}(a^2+b^2-c^2)(a^2-b^2+c^2)(-a^2+b^2+c^2). \tag{4}$$

A synthetic proof of (4) can be found in [20, p. 101]. Instead we will give an algebraic proof of the following more general result, using a technique outlined in [13].

The edges of a reversible tetrahedron $T$ come in (at most) 4 lengths. To see this, label the edge lengths of $T$ so that the triangular facets $f_1 \cong f_2$ have edge lengths $a, b, c$, with common edge of length $c$. Since $f_3 \cong f_4$, they must have edge lengths $a, b, d$. The six edges of $T$ then have lengths $a, a, b, b, c, d$, as in Figure 1.

**Theorem 3.1** (Volume Formula)**.** *Suppose that $T$ is a reversible tetrahedron having edge lengths $a, a, b, b, c, d$. Then*

$$V(T)^2 = \frac{1}{72}\Big(c^2d^2 - (a^2-b^2)^2\Big)\Big(a^2+b^2-\frac{c^2+d^2}{2}\Big). \tag{5}$$

The first polynomial factor in the formula (5) is a difference of two squares, so that (5) can be reformulated as

$$V(T)^2 = \frac{1}{72}(cd+a^2-b^2)(cd-a^2+b^2)\Big(a^2+b^2-\frac{c^2+d^2}{2}\Big). \tag{6}$$

In the special case where $c = d$, the tetrahedron $T$ is isosceles, and the formula (6) reduces to (4).

[1]Piero della Francesca (1415-1492), an Italian painter and geometer of the early Renaissance period.

The proof of (5) will make use of two identities from plane geometry. The well-known *parallelogram law* asserts that if edges of a parallelogram in $\mathbb{R}^2$ are labelled as in Figure 1, then $2a^2 + 2b^2 = c^2 + d^2$.

The less well-known *trapezoid law* asserts that, if the edges of a convex isosceles trapezoid are labelled as in Figure 3, then $b^2 - a^2 = cd$.

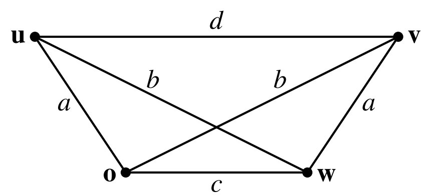


FIGURE 3. The trapezoid law: $b^2 - a^2 = cd$.

To see why, observe that

$$\begin{aligned} b^2 - a^2 &= |\mathbf{u} - \mathbf{w}|^2 - |\mathbf{v} - \mathbf{w}|^2 \\ &= \mathbf{u} \cdot \mathbf{u} - 2\mathbf{u} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{w} - \mathbf{v} \cdot \mathbf{v} + 2\mathbf{v} \cdot \mathbf{w} - \mathbf{w} \cdot \mathbf{w} \\ &= |\mathbf{u}|^2 - |\mathbf{v}|^2 + 2\mathbf{w} \cdot (\mathbf{v} - \mathbf{u}) \\ &= a^2 - b^2 + 2cd, \end{aligned}$$

where the last step follows from the parallelism of $\mathbf{w}$ and $\mathbf{v} - \mathbf{u}$. The trapezoid law now follows.

*Proof of The Volume Formula 3.1.* Let $f(a, b, c, d)$ denote the polynomial $V(T)^2$. The factors of $f$ can be determined by considering the cases in which the volume of $T$ is zero, namely, when the tetrahedron $T$ is flat or otherwise degenerate. If $T$ is reversible, this can occur in two ways.

In one case, $T$ may flatten to a parallelogram, having edges of length $a, b, a, b$ and diagonals of length $c, d$. In this instance, the parallelogram law for the standard inner product implies that $2a^2 + 2b^2 = c^2 + d^2$.

In the second case, $T$ may flatten to a trapezoid, having non-parallel edges of length $a, a$, parallel edges of length $c, d$, and diagonals of length $b, b$. In this instance, the trapezoid law implies that $(b^2 - a^2)^2 = c^2 d^2$,

These cases suggest both $2a^2 + 2b^2 - c^2 - d^2$ and $c^2d^2 - (b^2 - a^2)^2$ as possible factors of the polynomial $f$.

Denote $A = a^2$, $B = b^2$, $C = c^2$ and $D = d^2$. We observed following (3) above that $f$ is a polynomial in the *squared* values $a^2, b^2, c^2, d^2$, so that $f = f(A, B, C, D) \in \mathbb{R}[A, B, C, D]$. Since volume $V$ is homogeneous of degree 3 with respect to length, the polynomial $f = V^2$ is homogeneous of degree 6 with respect to the variables $a, b, c, d$, and is therefore homogeneous of degree 3 with respect to the variables $A, B, C, D$; that is, a homogeneous *cubic* polynomial in $\mathbb{R}[A, B, C, D]$.

To verify that $2a^2+2b^2-c^2-d^2$ is indeed a factor of $f(a,b,c,d)$, use division with remainder in $\mathbb{R}[A,B,C,D]$ to obtain

$$f(A,B,C,D) = (2A+2B-C-D)g(A,B,C,D) + r(B,C,D),$$

for some $g \in \mathbb{R}[A,B,C,D]$ and $r \in \mathbb{R}[B,C,D]$. Here division with remainder in $\mathbb{R}[A,B,C,D]$ is performed here using lexicographical order on the variables $A,B,C,D$. (See, for example, [6, p. 54].)

Note that $A$ does not appear in the polynomial expression for $r$. Suppose that $C > D > 0$. By the triangle inequality, each $B$ such that

$$\sqrt{C} - \sqrt{D} < 2\sqrt{B} < \sqrt{C} + \sqrt{D}$$

gives rise to a parallelogram as in Figure 4, yielding $A \geq 0$ so that $2A+2B-C-D = 0$. This degenerate reversible tetrahedron $T$ has volume zero, so that $f(A,B,C,D) = V^2 = 0$. It follows that $r(B,C,D) = 0$ on a non-empty open set. Since $r$ is a polynomial, it follows that $r$ is identically zero, so that

$$f(A,B,C,D) = (2A+2B-C-D)g(A,B,C,D).$$

In other words, $2A+2B-C-D$ divides $f$ in $\mathbb{R}[A,B,C,D]$.

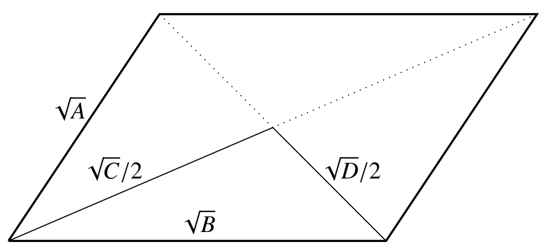


FIGURE 4. This parallelogram exists iff $\frac{1}{2}\sqrt{C} - \frac{1}{2}\sqrt{D} \leq \sqrt{B} \leq \frac{1}{2}\sqrt{C} + \frac{1}{2}\sqrt{D}$.

For the trapezoidal factors, view $f$ as polynomial in $\mathbb{R}[A,B,c,d]$, and write

$$f(A,B,c,d) = (cd-B+A)\tilde{g}(A,B,c,d) + \tilde{r}(B,c,d),$$

using division with remainder in $\mathbb{R}[A,B,c,d]$ under lexicographical order on the variables $A,B,c,d$. Once again the remainder $\tilde{r}$ is independent of the variable $A$, while a trapezoidal degenerate (zero volume) tetrahedron can be constructed for an open set of values $(B,c,d)$, so that $\tilde{r}$ is also identically zero. Therefore, $(cd-B+A)$ is also a factor $f$.

Finally, a symmetrical argument (reversing the roles of $A$ and $B$) yields a factor of $(cd-A+B)$.

Since $\mathbb{R}[A,B,c,d]$ is a unique factorization domain [2, p. 371][6, p. 149], the irreducible factors $(cd-B+A)$, and $(cd-A+B)$ are prime, so that

$$(cd-B+A)(cd-A+B) \;=\; c^2d^2-(B-A)^2 \;=\; CD-(B-A)^2$$

divides $f$.

Similarly, since $\mathbb{R}[A, B, C, D]$ is a unique factorization domain, the two irreducible factors $CD - (B - A)^2$, and $2A + 2B - C - D$ are prime in $\mathbb{R}[A, B, C, D]$, so that

$$V^2 = f = (2A + 2B - C - D)(CD - (B - A)^2)k. \tag{7}$$

Because $f$ is a homogeneous cubic polynomial in $\mathbb{R}[A, B, C, D]$, the factor $k$ must be a constant, independent of the parameters $A, B, C, D$.

To compute the constant $k$, recall that the volume of the regular (equilateral) tetrahedron of unit edge length $A = B = C = D = 1$ is $\sqrt{2}/12$. It follows that

$$\frac{1}{72} = \left(\frac{\sqrt{2}}{12}\right)^2 = V^2 = f(1, 1, 1, 1) = 2k.$$

Hence, $k = 1/144$, and (7) becomes (5). □

I. Izmestiev has pointed out that applying the *Regge symmetry* [1] to a reversible tetrahedron gives a new reversible tetrahedron having the same volume, and for which the factors of the Cayley-Menger polynomial (6) are permuted [12].

## 4. **Generalizations**

A convex polytope $P$ in $\mathbb{R}^n$ will be called *reversible* if there is an affine plane $\xi$ of codimension 2 such that $P$ is symmetric under the 180° rotation of the 2-plane $\xi^\perp$ that fixes $\xi$.

If a tetrahedron $T$ is $\mathbb{R}^3$ is symmetric under a 180° rotation around a line $\ell$, then this rotation must map facets to facets and facet normals to facet normals. In view of Proposition 1.1, the only way this can occur is when $\ell$ passes through the midpoints of two non-adjacent edges of $T$, so that $T$ must have pairs of congruent facets, as in the examples addressed earlier. It follows that this more general definition of a reversible polytope is consistent with the definition given earlier for tetrahedra in $\mathbb{R}^3$. However, naive analogues of the theorems of this paper do not follow, because this level of symmetry admits many more variations in structure for dimensions $n \geq 4$. Indeed, there exist 4-dimensional simplices in which all 5 facets have the same volume in spite of not being mutually congruent. For an extensive treatment of this subject, see [16].

In addition to admitting the Heron-type formula (4) for volume, isosceles tetrahedra satisfy many other characteristic properties (see, for example, [10, p. 90-97][14]). It would be interesting to consider what parallels these other properties may have in the more general context of reversible tetrahedra.

## References

[1] A. Akopyan and I. Izmestiev, The Regge symmetry, confocal conics, and the Schläfli formula, *Bull. London Math. Soc.*, **51** (2019), 765–775.
[2] M. Artin, *Algebra, 2nd ed.*, Prentice-Hall, Upper Saddle River, NJ, 2010.
[3] T. Bonnesen and W. Fenchel, *Theory of Convex Bodies*, BCS Associates, Moscow, Idaho, 1987.
[4] H. Coxeter, *Regular Polytopes*, Dover, New York, 1973.
[5] H. Coxeter and S. Greitzer, *Geometry Revisited*, MAA, Washington, D.C., 1967.
[6] D. Cox, J. Little, and D. O'Shea, *Ideals, Varieties, and Algorithms, 2nd ed.*, Springer, New York, 1996.

[7] C. D'Andrea and M. Sombra, The Cayley-Menger determinant is irreducible for $n \geq 3$, *Sib. Math. J.*, **46** (2005), 71–76.
[8] P. Fowler and A. Rasset, Is There a "Most Chiral Tetrahedron"?, *Chem. Eur. J.*, **10** (2004), 6575–6580.
[9] P. Fowler and A. Rasset, A classification scheme for chiral tetrahedra, *C. R. Chimie*, **9** (2006), 1203–1208.
[10] R. Honsberger, *Mathematical Gems II*, MAA, Washington, D.C., 1976.
[11] J. Horváth, A property of tetrahedra with equal faces in spaces of constant curvature (Hungarian, German summary), *Mat. Lapok*, **20** (1969), 257–263.
[12] I. Izmestiev. *Private communication*, (2022).
[13] D. Klain, An intuitive derivation of Heron's formula, *Amer. Math. Monthly*, **111** (2004), no. 8, 709–712.
[14] J. Leech, Some properties of the isosceles tetrahedron, *Math. Gaz.*, **34** (1950), no. 310, 269–271.
[15] H. Martini, Regular simplices in spaces of constant curvature, *Amer. Math. Monthly*, **100** (1993), no. 2, 169–171.
[16] P. McMullen, Simplices with equiareal faces, *Discrete Comput. Geom.*, **24** (2000), 397–411.
[17] M. Peterson, The geometry of Piero della Francesca, *Math. Intell.*, **19** (1997), no. 3, 33–40.
[18] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory, 2nd ed.*, Cambridge University Press, New York, 2014.
[19] D. Sommerville, *An Introduction to the Geometry of n Dimensions*, Dover, New York, 1958.
[20] H. Steinhaus, *One Hundred Problems in Elementary Mathematics*, Dover, New York, 1979.
[21] K. Wirth and A. Dreiding, Edge lengths determining tetrahedrons, *Elem. Math.*, **64** (2009), 160–170.
[22] K. Wirth and A. Dreiding, Tetrahedron classes based on edge lengths, *Elem. Math.*, **68** (2013), 56–64.
[23] K. Wirth and A. Dreiding, Relations between edge lengths, dihedral, and solid angles in tetrahedra, *J. Math. Chem.*, **52** (2014), 1624 –1638.

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