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Kinematic formulas for finite vector spaces

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Abstract

We derive q-analogues of some fundamental theorems of convex geometry, including Helly's theorem, the principal kinematic formula, and Hadwiger's characterization theorem for invariant valuations.

The essential link between convex geometry and combinatorial theory is the lattice structure of the collection of polyconvex sets; that is, the collection of all finite unions of compact convex sets in $\mathbb{R}^n$. This connection was highlighted by Rota in [16], where a valuation characterization theorem and kinematic formula were derived for the Boolean algebra of subsets of a finite set (see also [10]). In the present note we pursue this theme in the context of finite vector spaces.

To begin, we review a few well-known theorems of convex geometry, whose combinatorial analogues are developed in the sections following.

Helly's theorem gives a simple condition under which a finite collection of convex sets is guaranteed to have non-empty intersection [3, 4, 6, 8].

Theorem 0.1 (Helly's theorem). *Let F be a finite family of compact convex sets in $\mathbb{R}^n$. Suppose that, for any subset $G \subseteq F$ such that $|G| \leqslant n+1$ (that is, every subset of cardinality at most $n+1$ of F),*

$$\bigcap_{K \in G} K \neq \emptyset.$$

Then

$$\bigcap_{K \in F} K \neq \emptyset.$$

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In other words, if every $n+1$ elements of F have nonempty intersection, then the entire family F of convex sets has nonempty intersection.

Denote by $\mathscr{K}^n$ the set of all compact convex sets in $\mathbb{R}^n$. The set $\mathscr{K}^n$ is endowed with the topology induced by the *Hausdorff metric* on compact sets in $\mathbb{R}^n$ (see [18]). A function $\varphi : \mathscr{K}^n \to \mathbb{R}$ is called a *valuation* on $\mathscr{K}^n$ if $\varphi(\emptyset)=0$, where $\emptyset$ is the empty set, and

$$\varphi(K\cup L)=\varphi(K)+\varphi(L)-\varphi(K\cap L) \tag{1}$$

for all $K,L\in\mathscr{K}^n$ such that $K\cup L\in\mathscr{K}^n$ as well. A valuation φ on $\mathscr{K}^n$ is said to be *rigid motion invariant* if $\varphi(gK)=\varphi(K)$ for all rigid motions (translations, rotations, and reflections) g of $\mathbb{R}^n$.

Of particular interest are McMullen's *intrinsic volumes* [12; 13; 18, p. 210], which give invariant extensions of i-dimensional volume (on i-planes) to polyconvex subsets of $\mathbb{R}^n$, where $n \geqslant i$. Denote by $G(n,i)$ the set of all i-dimensional subspaces of $\mathbb{R}^n$, equipped with the invariant (Haar) measure v_i normalized so that

$$v_i(G(n,i))=\binom{n}{i}\frac{\omega_n}{\omega_i\omega_{n-i}},$$

where ω_i is the i-dimensional volume of the unit ball in $\mathbb{R}^i$. Denote by V_i the i-dimensional volume in $\mathbb{R}^i$. The i-volume V_i is extended to i-th intrinsic volume (also denoted V_i) on all of $\mathscr{K}^n$ by

$$V_i(K)=\int_{G(n,i)} V_i(K|\xi)\,\mathrm{d}v_i,$$

where $K|\xi$ denotes the orthogonal projection of K onto the subspace ξ.

The valuation V_0, which takes the value 1 on all non-empty compact convex sets, extends to the Euler characteristic on the lattice of polyconvex sets (see, for example, [10, 13, 18]).

Hadwiger's volume theorem states that V_n is the only continuous rigid motion invariant valuation on $\mathscr{K}^n$ that vanishes on compact convex sets of dimension less than n; i.e., on sets with empty interior. This theorem is easily shown to be equivalent to the following [7, 9, 10, 18]:

Theorem 0.2 (Hadwiger's characterization theorem). *Suppose that φ is a continuous rigid motion invariant valuation on $\mathscr{K}^n$. Then there exist $c_0,c_1,\ldots,c_n\in\mathbb{R}$ such that*

$$\varphi(K)=\sum_{i=0}^{n} c_i V_i(K)$$

for all $K\in\mathscr{K}^n$.

The last theorem from classical convex geometry for which we present a q-analogue is the *principal kinematic formula* and its variations.

Denote by E_n the group of rigid motions of $\mathbb{R}^n$; that is, the indirect sum of the translations group of $\mathbb{R}^n$ with the orthogonal group $O(n)$.

Theorem 0.3 (Principal kinematic formula). *For all polyconvex sets A and K,*

$$\int_{E_n} V_0(A \cap gK)\,\mathrm{d}g = \sum_{i=0}^{n} \binom{n}{i}^{-1} \frac{\omega_i \omega_{n-i}}{\omega_n} V_i(A) V_{n-i}(K). \tag{2}$$

The integral in (2) is taken with respect to the indirect sum of the Lebesgue measure on $\mathbb{R}^n$ with the Haar probability measure on $O(n)$. For compact *convex* sets A and K this integral has an evident geometric interpretation as the measure of the set of $g \in E_n$ such that $A \cap gK \neq \emptyset$. Alternatively one may think of (2) as the 'measure' of all convex sets gK in $\mathbb{R}^n$ congruent to K that meet A.

Theorem 0.3 is one of a family of kinematic formulas for valuations on polyconvex sets, variously attributed in origin to Blaschke [1], Chern [2], and Santaló [17]. The techniques of the present work are inspired by those of Hadwiger [7] and Rota [16] (also [10]). Kinematic formulas remain a topic of current interest in convex and integral geometry; see [10, 18, 19, 21].

1. The subspace poset

In this section we recall some well-known facts about finite vector spaces.

Let $\boldsymbol{F}$ be a finite field having q elements, where q is a positive power of a prime number, and let V be vector space over $\boldsymbol{F}$ of dimension n. Denote by $L(V)$ the partially ordered set of subspaces of V, a (finite) poset ordered by inclusion $\subseteq$. It is well known that vector sum and intersection of subspaces coincide with least upper bound and greatest lower bound in the partially ordered set $L(V)$. We denote the elements of $L(V)$ by lower case letters x, y, etc.

A *segment* of $L(V)$, denoted by $[x, y]$, where $x \leqslant y$, consists of all elements $z \in L(V)$ such that $x \leqslant z \leqslant y$. Every segment $[x, y]$ is isomorphic to the poset $L(\boldsymbol{F}^{(\dim y - \dim x)})$.

A *chain* in $L(V)$ is a linearly ordered subset. A *flag* F in $L(V)$ is a maximal chain; that is, a chain such that if $G \supseteq F$ and G is a chain, then $G = F$. An *antichain* is a subset $A \subseteq L(V)$ such that if $x, y \in A$ then neither $x \subset y$ nor $y \subset x$.

The antichain consisting of all elements of $L(V)$ of dimension k shall be denoted by $L_k(V)$. The size, or number of elements, of $L_k(V)$ is the *q-binomial coefficient* (also called the *Gaussian coefficient*), denoted

$$\binom{n}{k}_q.$$

The number $N(V)$ of automorphisms of V (i.e., bijective linear maps from V to itself) will be of use in the sections following. To compute $N(V)$, choose a basis $s_1, s_2, \ldots, s_n$ for V, and suppose that $g: V \to V$ is an automorphism. In this case, there are q^n-1 possible values for $g(s_1)$. Having assigned $g(s_1)$, there remain q^n-q possible values for $g(s_2)$, etc. Proceeding in this manner, we obtain

$$N(V)=(q^n-1)(q^n-q)\cdots(q^n-q^{n-1}).$$

In a similar manner we compute the number $N(V,k)$ of automorphisms g of V that fix a given subspace $x\in L_k(V)$, i.e., such that $gx=x$. Once again choose a basis $s_1,s_2,\ldots,s_n$ for V, this time so that $s_1,\ldots,s_k$ is a basis for x. Since $g(s_i)\in x$ for $1\leqslant i\leqslant k$, there are

$$(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})$$

possible assignments for the first k basis elements, i.e., values for $g(s_1),\ldots,g(s_k)$. Meanwhile, there are q^n-q^k remaining possible values for $g(s_{k+1})$, and then q^n-q^{k+1} remaining possible values for $g(s_{k+2})$, etc., so that

$$N(V,k)=(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})(q^n-q^k)(q^n-q^{k+1})\cdots(q^n-q^{n-1}).$$

In order to compute the q-binomial coefficient, note that if $x\in L_k(V)$, then $gx\in L_k(V)$ as well, for *any* automorphism g of V. Moreover, for $x,y\in L_k(V)$ there are exactly $N(V,k)$ automorphisms g such that $gx=y$. It follows that

$$\binom{n}{k}_q=\frac{N(V)}{N(V,k)}.$$

In other words,

$$\binom{n}{k}_q=\frac{(q^n-1)(q^n-q)\cdots(q^n-q^{k-1})}{(q^k-1)(q^k-q)\cdots(q^k-q^{k-1})}=\frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)}. \tag{3}$$

For positive integers k, denote

$$[k]_q=\frac{q^k-1}{q-1}=1+q+\cdots+q^{k-1},$$

and denote

$$[k]_q!=[k]_q[k-1]_q\cdots[1]_q.$$

By dividing out common factors of $q-1$ in the numerator and denominator of (3), we obtain

$$\binom{n}{k}_q=\frac{[n]_q!}{[k]_q![n-k]_q!}. \tag{4}$$

Notice that as $q\to 1$ in (4) we have $\binom{n}{k}_q\to\binom{n}{k}$, the classical binomial coefficient.

For a thorough treatment of the combinatorial theory of finite vector spaces, see [5, 11]. See also [20, pp. 126–127].

2. A q-analogue of Helly's theorem

Our first q-analogue will be a Helly-type theorem for subspaces of a vector space V of dimension n over $\boldsymbol{F}$.

Theorem 2.1 (Helly's theorem for subspaces). *Let F be a finite family of subspaces of V. Suppose that, for any subset $G \subseteq F$ such that $|G| \leqslant n$ (that is, every subset of cardinality at most n of F),*

$$\dim\left(\bigcap_{V\in G} V\right) > 0.$$

Then

$$\dim\left(\bigcap_{V\in F} V\right) > 0.$$

In other words, if every n elements of F contain a common line through the origin, then there is (at least one) line ℓ contained in all of the subspaces in F. This theorem actually follows easily from elementary linear algebra, independently of the field $\boldsymbol{F}$.

Proof. The proof is by induction on the size $|F|$ of the family F. If $|F| \leqslant n$ then the theorem holds trivially. Suppose the theorem holds for the case $|F| = m$ for some $m \geqslant n$. We then consider the case of $|F| = m+1$.

Write $F = \{x_1, x_2, \ldots, x_{m+1}\}$, and denote

$$y_i = \bigcap_{j \neq i} x_j. \tag{5}$$

Since the theorem is true for families of size m, each y_i has positive dimension. That is, for each $i \in \{1, \ldots, m+1\}$ there exists a non-zero vector $w_i \in y_i$. Since $m \geqslant n$, the collection $\{w_1, w_2, \ldots, w_{m+1}\}$ must be linearly dependent. Without loss of generality, we may then assume that

$$w_{m+1} = c_1 w_1 + \cdots + c_m w_m,$$

where not all the coefficients c_i are zero. But (5) implies that $w_i \in x_{m+1}$ for all $i \in \{1, \ldots, m\}$. It follows that $w_{m+1} \in x_{m+1}$ as well. Since $w_{m+1} \in y_{m+1}$, we have

$$w_{m+1} \in \bigcap_{j=1}^{m+1} x_j.$$

This completes the induction step and the proof of Theorem 2.1. $\square$

As was previously noted, the proof of Theorem 2.1 applies to finite families of subspaces of a finite-dimensional vector space over *any* field, not only finite fields. If we replace the field $\boldsymbol{F}$ with the real numbers $\mathbb{R}$ (or the complex numbers $\mathbb{C}$), then Theorem 2.1 can be shown (by means of a standard compactness argument) to hold for any (possibly infinite) family F of subspaces satisfying the n-intersection condition.

3. Valuations on the lattice of order ideals

Define an *order ideal* to be a subset A of $L(V)$ such that, if $x\in A$ and $y\leqslant x$ then $y\in A$. An order ideal is a partially ordered set in the induced order of $L(V)$. The set of maximal elements of an order ideal is an antichain. An order ideal having exactly one maximal element is called a *simplex* or a *principal ideal.*

The (set-theoretic) union and intersection of any number of order ideals is again an order ideal. Thus, the set $J(V)$ of all order ideals in $L(V)$ is a distributive lattice, and we can study valuations on $J(V)$.

A function $\varphi : J(V)\to\mathbb{R}$ is called a *valuation* if $\varphi(\{0\})=0$, and

$$\varphi(A\cup B)+\varphi(A\cap B)=\varphi(A)+\varphi(B)$$

for all $A,B\in J(V)$.

For $x\in L(V)$, denote by $\bar{x}$ the simplex whose maximal element is x; that is, the set of all $y\in L(V)$ such that $y\leqslant x$.

It is well-known (or see [10, 15]) that every valuation φ on $J(V)$ extends uniquely to a valuation, again denoted by φ, on the Boolean algebra $P(L(V))$ of all subsets of $L(V)$, which is generated by $J(V)$. Such a valuation is evidently determined by its value on the one element subsets of $P(L(V))$; that is, by arbitrarily assigning a value $\varphi(\{x\})$ for each $x\in L(V)$.

Let x be of dimension k, and let $A_1,A_2,\dots,A_{[k]_q}$ be the maximal simplices $A_i\subseteq\bar{x}$ such that $A_i\neq\bar{x}$. Then

$$\varphi(\{x\})=\varphi(\bar{x})-\varphi(A_1\cup A_2\cup\cdots\cup A_{[k]_q}).$$

The right-hand side can be computed in terms of simplices of lower dimension, by the inclusion–exclusion principle (1). Thus, by induction on the dimension, we have the following theorem (see also [15]).

Theorem 3.1. *Every valuation φ on the distributive lattice $J(V)$ of all order ideals is uniquely determined by the values $\varphi(\bar{x})$, $x\in L(V)$. The values $\varphi(\bar{x})$ may be arbitrarily assigned.*

A valuation φ on $J(V)$ is called *invariant* if it is invariant under the group $GL(V)$ of automorphisms of the vector space V; that is, if $\varphi(A)=\varphi(gA)$ for every order ideal A and for every linear isomorphism $g: V\to V$ (which induces an action on $J(V)$, also denoted by g).

Next, we establish the existence of the Euler characteristic. The following is an immediate consequence of Theorem 3.1.

Theorem 3.2. *There exists a unique invariant valuation φ on $J(V)$, called the Euler characteristic, such that $\chi(\bar{x})=1$ for every simplex $\bar{x}$ with* $\dim(x)>0$, *and such that* $\chi(\overline{\{0\}})=0$.

Recall that the Euler characteristic of the distributive lattice J of order ideals of a poset P is given by

$$\chi(A) = -\sum_{x\in A,\, x>0} \mu(0,x), \tag{6}$$

where 0 denotes the minimal element of P and μ is the *Möbius function* of P (see [15; 11; 20, p. 120]). The Möbius function of the poset $L(V)$ is given by

$$\mu(0,x) = (-1)^{\dim(x)} q^{\binom{\dim(x)}{2}}. \tag{7}$$

For a derivation of (7), see [5, 14, 11] or [20, pp. 126–127].

For $i>0$, set

$$\varphi_i(\bar{x}) = |\bar{x} \cap L_i(V)|,$$

and extend φ_i to all of $J(V)$ by Theorem 3.1. For every order ideal A,

$$\varphi_i(A) = |A \cap L_i(V)|.$$

In other words, the valuation φ_i counts the number of i-subspaces of V contained in an order ideal. If $x \in L_k(V)$ then the order ideal $(\bar{x})$ is isomorphic (as a lattice) to $L(x)$, so that

$$\varphi_i(\bar{x}) = \binom{k}{i}_q. \tag{8}$$

Combining (6) and (7) now yields

$$\chi(A) = \sum_{k=1}^{n} \sum_{x\in A,\, \dim(x)=k} (-1)^{k+1} q^{\binom{k}{2}} = \sum_{k=1}^{n} (-1)^{k+1} q^{\binom{k}{2}} \varphi_k(A) \tag{9}$$

for any order ideal A.

More generally, we have the following q-analogue of Hadwiger's characterization theorem for invariant valuations.

Theorem 3.3 (q-basis theorem). *The invariant valuations $\varphi_1, \ldots, \varphi_n$ span the vector space of all invariant valuations φ on $J(V)$.*

Proof. Suppose that φ is an invariant valuation on $J(V)$. Extend φ to all of $P(L(V))$. The extended valuation, which is still denoted φ, is again invariant. If x and y have the same dimension in $L(V)$, then there exists an automorphism g of V such that $gx=y$. Therefore, $\varphi(\{x\}) = \varphi(\{y\}) = c_i$, for some constant c_i. Thus, the valuation

$$\varphi - \sum_{i=1}^{n} c_i \varphi_i$$

vanishes on all singleton sets $\{x\}$, for all $x \in L(V)$, and consequently vanishes on all of $P(L(V))$. $\square$

In order to compute the coefficients c_i given by the q-Basis Theorem 3.3, note that if $x_k \in L_k(V)$, then

$$\varphi(\{x_k\}) = \sum_{i=1}^{n} c_i \varphi_i(\{x_k\}) = c_k. \tag{10}$$

If we know the values of $\varphi(\{x_k\})$ for each $k = 1, \ldots, n$, we are done.

However, a valuation φ is often given in terms of its values on simplices $\bar{x}$, for $x \in L(V)$ (as in, for example, Theorem 3.1). In order to compute the values $\varphi(\{x\})$, given the values $\varphi(\bar{x})$, we use Möbius inversion. Recall that the extension of a valuation φ on $J(V)$ to all of $P(L(V))$ is given inductively by

$$\varphi(\{0\}) = 0$$

and

$$\varphi(\{x\}) = \varphi(\bar{x}) - \sum_{y<x} \varphi(\{y\}),$$

so that

$$\varphi(\bar{x}) = \sum_{y \leqslant x} \varphi(\{y\}) \tag{11}$$

for all $x \in L(V)$. Applying Möbius inversion to (11) yields

$$\varphi(\{x\}) = \sum_{y \leqslant x} \mu(y,x)\varphi(\bar{y}) = \sum_{y \leqslant x} (-1)^{\dim(x)-\dim(y)} q^{\binom{\dim(x)-\dim(y)}{2}} \varphi(\bar{y}). \tag{12}$$

Combining (12) with (10) along with the invariance of φ, we obtain

$$\begin{aligned} c_k &= \varphi(\{x_k\}) \\ &= \sum_{y \leqslant x_k} (-1)^{k-\dim(y)} q^{\binom{k-\dim(y)}{2}} \varphi(\bar{y}), \end{aligned}$$

so that

$$c_k = \sum_{i=1}^{k} (-1)^{k-i} q^{\binom{k-i}{2}} \binom{k}{i}_q \varphi(\bar{x}_i) \tag{13}$$

for each $k = 1, \ldots, n$.

4. Kinematic formulas for $L(V)$

As an application of the q-Basis Theorem 3.3 we shall derive a q-analogue of the principal kinematic formula (2) for compact convex sets.

One way to construct invariant valuations on $J(V)$ is the following. Start with any valuation φ on $J(V)$ such that $\varphi(\{\emptyset\}) = 0$, and let B be any order ideal. For any order ideal A, define

$$\varphi(A; B) = \frac{1}{N(V)} \sum_{g} \varphi(A \cap gB),$$

where g ranges over all automorphisms of the vector space V. For fixed A, the set function $\varphi(A;B)$ is a valuation in the variable B; in fact, it is an invariant valuation, since

$$\varphi(A;g_0B)=\frac{1}{N(V)}\sum_g \varphi(A\cap gg_0B)$$
$$=\frac{1}{N(V)}\sum_g \varphi(A\cap gB)$$

for each automorphism g_0. By Theorem 3.3, the functional $\varphi(A;B)$ can be expressed as a linear combination of the valuations φ_i, with coefficients $c_i(A)$ depending on A:

$$\varphi(A;B)=\sum_{i=1}^{n} c_i(A)\varphi_i(B). \tag{14}$$

Meanwhile, for fixed B, the set function $\varphi(A;B)$ is a valuation in the variable A. From this it follows that each of the coefficients $c_i(A)$ is a valuation in the variable A.

Now consider the case when φ is an *invariant* valuation. If so, then

$$\varphi(A;B)=\frac{1}{N(V)}\sum_g \varphi(A\cap gB)=\frac{1}{N(V)}\sum_g \varphi(g^{-1}A\cap B)$$
$$=\frac{1}{N(V)}\sum_g \varphi(gA\cap B)=\varphi(B;A).$$

Therefore,

$$\varphi(A;B)=\sum_{i,j=1}^{n} c_{ij}\varphi_i(A)\varphi_j(B).$$

Since $\varphi(A;B)=\varphi(B;A)$, it is evident that $c_{ij}=c_{ji}$. It turns out that most of the constants c_{ij} are equal to zero. To compute the coefficients c_{ij}, extend the valuation φ to the Boolean algebra $P(L(V))$ generated by $J(V)$, and let α_i denote the value of φ on a singleton set in $P(L(V))$ whose single element is an i-dimensional subspace of V (that is, $\alpha_i=\varphi(\{x_i\})$, for any $x_i\in L_i(V)$).

Theorem 4.1 (General q-kinematic formula). *Suppose that φ is an invariant valuation on $J(V)$. For all $A,B\in J(V)$,*

$$\varphi(A;B)=\sum_{i=1}^{n}\binom{n}{i}_q^{-1}\alpha_i\varphi_i(A)\varphi_i(B). \tag{15}$$

Recall that the values α_i may be obtained from (10) or (13).

Proof. Suppose that $x_i, y_j\in L(V)$ of dimension i and j respectively. Let $A=\{x_i\}$ and $B=\{y_j\}$. For any automorphism g of V, the set $A\cap gB=\emptyset$ if $i\neq j$. If $i=j$ then

$A \cap gB = \emptyset$ if $x_i \neq gy_j$, and there are $N(V,i)$ automorphisms g of V such that $x_i = gy_j$. Hence, we have

$$\varphi(A;B) = \frac{1}{N(V)} \sum_g \varphi(A \cap gB) = \frac{N(V,i)}{N(V)} \varphi(A) = \binom{n}{i}_q^{-1} \alpha_i.$$

Meanwhile, $\varphi_k(A) = 1$ if $k = i$ and is otherwise equal to zero. Similarly, $\varphi_k(B) = 1$ if $k = j$ and is otherwise equal to zero. Hence,

$$\varphi(A;B) = \sum_{i,j=1}^{n} c_{ij} \varphi_i(A) \varphi_j(B) = c_{ij}.$$

Therefore, $c_{ij} = \binom{n}{i}_q^{-1} \alpha_i$ if $i = j$ and is otherwise equal to zero. □

The case $\varphi = \chi$ is of particular interest. The Euler formula (9) implies that $\chi(\{x_i\}) = (-1)^{i+1} q^{\binom{i}{2}}$. Theorem 4.1 then specializes to the following q-analogue of Theorem 0.3.

Theorem 4.2 (Principal q-kinematic formula). *For all* $A, B \in J(V)$,

$$\frac{1}{N(V)} \sum_g \chi(A \cap gB) = \sum_{i=1}^{n} (-1)^{i+1} q^{\binom{i}{2}} \binom{n}{i}_q^{-1} \varphi_i(A) \varphi_i(B). \tag{16}$$

The probability that a randomly chosen l-simplex $\bar{y}_l$ shall meet a fixed k-simplex $\bar{x}_k$; i.e., that a randomly chosen l-subspace y_l meets a fixed k-subspace x_k with $\dim(x_k \cap y_l) > 0$; can now be computed by combining (8) and (16) to yield

$$\frac{1}{N(V)} \sum_g \chi(\bar{x}_k \cap g\bar{y}_l) = \sum_{i=1}^{n} (-1)^{i+1} q^{\binom{i}{2}} \binom{n}{i}_q^{-1} \binom{k}{i}_q \binom{l}{i}_q. \tag{17}$$

By combining Theorems 4.1 and 4.2 with some elementary probabilistic reasoning one can obtain polynomial identities. For example, if $k + l > n$, then $\dim(x_k \cap gy_l) > 1$ for all k-simplices $\bar{x}_k$, l-simplices $\bar{y}_l$, and automorphisms g of V. In other words, $\dim(x_k \cap gy_l) > 1$ with unit probability. It follows that

$$\sum_{i=1}^{n} (-1)^{i+1} q^{\binom{i}{2}} \binom{n}{i}_q^{-1} \binom{k}{i}_q \binom{l}{i}_q = 1, \tag{18}$$

whenever $k + l > n$.

For the case of $k + l \leqslant n$, we compute instead the probability that $\dim(x_k \cap gy_l) = 0$, for a random automorphism g. Let $s_1, \ldots, s_n$ be a basis for V such that $s_1, \ldots, s_l$ is a basis for y_l. For $\dim(x_k \cap gy_l) = 0$ to hold, we require $gs_1 \in V - x_k$, of which there are $q^n - q^k$ choices. There then remain $q^n - q^{k+1}$ possible values for gs_2, etc., so that there are

$$(q^n - q^k)(q^n - q^{k+1}) \cdots (q^n - q^{k+l-1}),$$

choices of values for $gs_1, \ldots, gs_l$. Having chosen the values of $gs_1, \ldots, gs_l$, which span a space of dimension l, there remain $q^n - q^l$ possible values for gs_{l+1}, and then $q^n - q^{l+1}$

possible values for gs_{l+2}, and so on, up to $q^n - q^{n-1}$ possible values for gs_n. It follows that there are

$$(q^n - q^k)\cdots(q^n - q^{k+l-1})(q^n - q^l)\cdots(q^n - q^{n-1})$$

automorphisms g of V such that $\dim(x_k \cap gy_l) = 0$. Hence, the probability that $\dim(x_k \cap gy_l) = 0$ for a random automorphism g is

$$\begin{aligned}
&\frac{(q^n - q^k)\cdots(q^n - q^{k+l-1})(q^n - q^l)\cdots(q^n - q^{n-1})}{N(V)} \\
&= \frac{(q^n - q^k)\cdots(q^n - q^{k+l-1})(q^n - q^l)\cdots(q^n - q^{n-1})}{(q^n - 1)(q^n - q)\cdots(q^n - q^{n-1})} \\
&= q^{kl}\frac{(q^{n-k} - 1)(q^{n-k} - q)\cdots(q^{n-k} - q^{l-1})}{(q^n - 1)(q^n - q)\cdots(q^n - q^{l-1})} \\
&= q^{kl}\frac{(q^{n-k} - 1)(q^{n-k-1} - 1)\cdots(q^{n-k-l+1} - 1)}{(q^n - 1)(q^{n-1} - 1)\cdots(q^{n-l+1} - 1)} \\
&= q^{kl}\frac{[n-k]_q!}{[n-k-l]_q!}\frac{[n-l]_q!}{[n]_q!}\frac{[k]_q!}{[k]_q!} \\
&= q^{kl}\binom{n}{k}_q^{-1}\binom{n-l}{k}_q.
\end{aligned}$$

It then follows from (17) that

$$\sum_{i=1}^{n}(-1)^{i+1}q^{\binom{i}{2}}\binom{n}{i}_q^{-1}\binom{k}{i}_q\binom{l}{i}_q = 1 - q^{kl}\binom{n}{k}_q^{-1}\binom{n-l}{k}_q. \tag{19}$$

Note that (19) is consistent with (18), since

$$\binom{n-l}{k}_q = 0$$

whenever $k + l > n$. By adding the term corresponding $i = 0$ to both sides of (19) and multiplying by -1 we obtain the following reformulation.

Theorem 4.3.

$$\sum_{i=0}^{n}(-1)^{i}q^{\binom{i}{2}}\binom{n}{i}_q^{-1}\binom{k}{i}_q\binom{l}{i}_q = q^{kl}\binom{n}{k}_q^{-1}\binom{n-l}{k}_q, \tag{20}$$

for all positive integers n, q and all $0 \leqslant k,\ l \leqslant n$.

Remark. After setting $q = 1$ the formulas (15) and (16) reduce to discrete kinematic formulas for random simplices in the Boolean algebra of subsets of a finite set S, provided 'automorphisms' g are replaced with *permutations* on the elements of S. Eqs. (18)–(20) also reduce to analogous equations involving the classical binomial coefficients. For a detailed description, see [10].

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