On the Minkowski condition and divergence

Daniel A. Klain
Department of Mathematical Sciences
University of Massachusetts Lowell
Lowell, MA 01854 USA
Daniel_Klain@uml.edu

Abstract The Minkowski condition for convex polytopes is equivalent to the divergence theorem of classical multivariable calculus.

1 The Minkowski condition for polytopes

It is easy to see that a convex polygon in $\mathbb{R}^2$ is uniquely determined (up to translation) by the directions and lengths of its edges. This suggests the following, less easily answered, question in higher dimensions: given a collection of proposed facet normals and facet areas, is there a convex polytope in $\mathbb{R}^d$ whose facets fit the given data, and, if so, is the resulting polytope unique? This question, along with its answer, is known as the *Minkowski problem.*

A convex polytope in $\mathbb{R}^d$ is defined to be the convex hull of a finite set of points in $\mathbb{R}^d$. For a convex polytope $P \subseteq \mathbb{R}^d$ and a unit vector $u \in \mathbb{R}^d$, denote by P_u the orthogonal projection of P onto the hyperplane $u^\perp$, and denote by P^u the support set of P in the direction of u. Since P is a polytope, P^u will be a face of P. If $\dim P = n$ and P^u is a face of dimension $n-1$ then P^u is called a *facet* of P, where u is the corresponding *facet normal.* The volume of a polytope P will be denoted by $V(P)$. If P is a subset of a hyperplane in $\mathbb{R}^d$, denote the $(d-1)$-dimensional volume of P by $v(P)$.

More generally one could ask: Given a collection $u_1, \ldots, u_k$ of unit vectors and $\alpha_1, \ldots, \alpha_k > 0$, under what condition does there exist a polytope P having the u_i as its facet normals and the α_i as its facet areas; that is, such that $v(P^{u_i}) = \alpha_i$ for each i?

A necessary condition on the facet normals and facet areas is given by the following proposition [BF48, Sch93a].

Proposition 1.1 (Minkowski condition) *Suppose that a convex polytope P has facet normals $u_1, u_2, \ldots, u_k$ and corresponding facet areas $\alpha_1, \alpha_2, \ldots, \alpha_k$. Then*

$$\alpha_1 u_1 + \cdots + \alpha_k u_k = 0. \tag{1}$$

Proof No. 1: If $u \in \mathbb{R}^d$ is a unit vector, then $|u_i \cdot u|\alpha_i$ is equal to the area of the orthogonal projection of the i-th facet of P onto the hyperplane $u^\perp$. Summing over all facets whose outward normals form an acute angle with u we obtain

$$\sum_{u_i \cdot u > 0} (u_i \cdot u)\alpha_i = v(P_u),$$

where P_u denotes the orthogonal projection of P onto the hyperplane $u^\perp$. Meanwhile summing analogously over all facets whose outward normals form an obtuse angle with u yields the value $-v(P_u)$.

Let $w = \alpha_1 u_1 + \cdots + \alpha_k u_k$. It now follows that

$$w \cdot u = \sum_i (u_i \cdot u)\alpha_i = \sum_{u_i \cdot u > 0} (u_i \cdot u)\alpha_i + \sum_{u_i \cdot u < 0} (u_i \cdot u)\alpha_i = v(P_u) - v(P_u) = 0.$$

In other words, $w \cdot u = 0$ for all unit vectors u, so that $w = 0$. ∎

The reader may have noticed that the Minkowski condition (1) is also an immediate consequence of the classical divergence theorem of multivariable calculus. A proof along this line runs as follows.

Proof No. 2: Let $v \in \mathbb{R}^d$ be a non-zero vector, and let $w = \alpha_1 u_1 + \cdots + \alpha_k u_k$. Define a vector field $F : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ to be the constant $F(x) = v$. Then

$$v \cdot w = \sum_i (v \cdot u_i)\alpha_i = \int_{\partial P} F \cdot \hat{n} \, dS, \tag{2}$$

the surface (flux) integral of the vector field F across the boundary of P.

Since F is a constant vector field, it is divergenceless; that is, $\nabla \cdot F = 0$. Applying the divergence theorem to (2) now yields

$$v \cdot w = \int_P \nabla \cdot F \, dV = \int_P 0 \, dV = 0.$$

Since v is arbitrary, it follows that $w = 0$. ∎

In the next section we will show that Proposition 1.1 is in fact *equivalent* to the divergence theorem.

Proposition 1.1 illustrates a necessary condition for existence of a polytope having a given set of facet normals and facet areas. The fundamental importance of condition (1) was first observed by Minkowski, who also discovered that the *converse* of Proposition 1.1 (along with some minor additional assumptions) is also true. In other words, the condition (1) is both necessary and (almost) sufficient to determine a polytope in $\mathbb{R}^d$ that is unique up to translation congruence.

Theorem 1.2 (Minkowski Existence Theorem) *Suppose $u_1, u_2, \ldots, u_k \in \mathbb{R}^d$ are unit vectors that span $\mathbb{R}^d$, and suppose that $\alpha_1, \alpha_2, \ldots, \alpha_k > 0$. Then there exists a convex polytope P having facet unit normals $u_1, u_2, \ldots, u_k$ and corresponding facet areas $\alpha_1, \alpha_2, \ldots, \alpha_k$, if and only if*

$$\alpha_1 u_1 + \cdots + \alpha_k u_k = 0.$$

Moreover, if Q is a convex polytope sharing the same facet normals u_i and corresponding facet areas α_i, then there exists $x \in \mathbb{R}^n$ such that $Q = P + x$.

For a proof of this theorem (and its many generalizations) see any of [BF48, Lut93, Sch93a]. Once the surface data are suitably defined, the Minkowski Existence Theorem can also be generalized to the context of compact convex sets [Sch93a], to the p-mixed volumes of the Brunn-Minkowski-Firey theory [Lut93], and to electrostatic capacity [Jer96]. See also [Sch93b] for a extensive survey on the Minkowski Existence Theorem and its applications.

2 The divergence theorem

We will now use the Minkowski condition (1) to prove the divergence theorem. For simplicity of presentation, we will restrict our attention to flux integrals of C^1 vector fields across the boundary ∂P of a convex polytope P.

Theorem 2.1 *Let P be convex polytope in $\mathbb{R}^d$, and let $F : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a C^1 vector field defined in an open neighborhood of P. Then*

$$\int_{\partial P} F \cdot \hat{n} \, dS = \int_P \nabla \cdot F \, dV.$$

Proof: Let $\mathcal{T}_\epsilon$ be a tiling of $\mathbb{R}^d$ by d-dimensional cubes of side length $\epsilon > 0$. Decompose P into d-cells σ formed by the intersection of P with the cubes of $\mathcal{T}_\epsilon$. Note that the facets of each cell σ are parallel to the coordinate hyperplanes (having been inherited from a d-cube of $\mathcal{T}_\epsilon$) when not part of ∂P. For each d-cell σ in the decomposition of P, let σ^* denote the centroid of the unique cube of $\mathcal{T}_\epsilon$ that contains σ. For each facet τ of σ, let $n_{\sigma,\tau}$ denote the normal vector to τ pointing *outward* from σ and having length equal to the $(d-1)$-dimensional volume of τ.

Note that if τ lies in the boundary ∂P of P, then τ is a facet of *exactly one* d-cell σ, to be denoted σ_τ or just σ, when the context is clear. On the other hand, if τ does not lie in the boundary ∂P, then τ is parallel to a coordinate hyperplane and co-bounds *exactly two* d-cells, to be denoted $\sigma_{\tau+}$ (or σ_+, when the context is clear) and $\sigma_{\tau-}$ (or σ_-), corresponding to the positive and negative sides of τ, as determined by the coordinate axis of $\mathbb{R}^d$ that is parallel to $\tau^\perp$.

Consider the sum

$$\sum_{\substack{(\tau,\sigma)\\ \tau\subseteq\sigma\subseteq P}} F(\sigma^*)\cdot n_{\sigma,\tau},$$

where each σ is a d-cell of P and each τ is a facet of the corresponding cell σ. We will evaluate this sum in two different ways.

Summing first over d-cells $\sigma \subseteq P$ and then over each facet $\tau \subseteq \sigma$, we obtain

$$\sum_{\substack{(\tau,\sigma)\\ \tau\subseteq\sigma\subseteq P}} F(\sigma^*)\cdot n_{\sigma,\tau} = \sum_{\sigma\subseteq P}\sum_{\tau\subseteq\partial\sigma} F(\sigma^*)\cdot n_{\sigma,\tau} = \sum_{\sigma\subseteq P} F(\sigma^*)\cdot \sum_{\tau\subseteq\partial\sigma} n_{\sigma,\tau}.$$

Applying the Minkowski condition (1) to the boundary of the convex polyhedral cell σ yields

$$\sum_{\tau\subseteq\partial\sigma} n_{\sigma,\tau} = 0,$$

so that

$$\sum_{\substack{(\tau,\sigma)\\ \tau\subseteq\sigma\subseteq P}} F(\sigma^*)\cdot n_{\sigma,\tau} = \sum_{\sigma\subseteq P} F(\sigma^*)\cdot 0 = 0. \tag{3}$$

Meanwhile, summing first over $(d-1)$-dimensional cells $\tau \subseteq P$ and then over each d-cell $\sigma \supseteq \tau$ we obtain

$$\begin{aligned}
\sum_{\substack{(\tau,\sigma)\\ \tau\subseteq\sigma\subseteq P}} F(\sigma^*)\cdot n_{\sigma,\tau} &= \sum_{\tau}\sum_{\sigma\supseteq\tau} F(\sigma^*)\cdot n_{\sigma,\tau} \\
&= \sum_{\tau\subseteq\partial P} F(\sigma_\tau^*)\cdot n_{\sigma_\tau,\tau} + \sum_{\tau\nsubseteq\partial P} [F(\sigma_{\tau+}^*)\cdot n_{\sigma_+,\tau} + F(\sigma_{\tau-}^*)\cdot n_{\sigma_-,\tau}]
\end{aligned}$$

$$= \sum_{\substack{\tau\subseteq\partial P\\ \sigma=\delta\tau}} F(\sigma^*)\cdot n_{\sigma,\tau} - \sum_{\substack{\tau\nsubseteq\partial P\\ \tau=\sigma_+\cap\sigma_-}} [F(\sigma_+^*) - F(\sigma_-^*)]\cdot n_{\sigma_-,\tau}, \tag{4}$$

since $n_{\sigma_+,\tau} = -n_{\sigma_-,\tau}$.

Recall that $\sigma = \delta\tau$ (σ is the co-boundary of τ) in the first sum of (4) because τ lies on the boundary of P in this sum, so that τ lies on the boundary of exactly one d-cell σ of P. The first sum of (4) approximates the surface integral of F across the boundary ∂P; that is,

$$\lim_{\epsilon\to 0} \sum_{\substack{\tau\subseteq\partial P\\ \sigma=\delta\tau}} F(\sigma^*)\cdot n_{\sigma,\tau} \;=\; \int_{\partial P} F\cdot\hat{n}\; dS. \tag{5}$$

Similarly, the second sum of (4) approximates the integral of the divergence $\nabla \cdot F$ over the volume of P. To see this, note that each interior τ is the facet of a d-cube with normal parallel to a coordinate axis. In other words, $n_\tau = \epsilon^{d-1}\hat{e}_i$, where $\hat{e}_i$ is the i-th unit coordinate vector for $\mathbb{R}^d$ and ϵ^{d-1} is the $(d-1)$-dimensional volume of τ. Hence,

$$\sum_{\substack{\tau \not\subseteq \partial P \\ \tau = \sigma_+ \cap \sigma_-}} [F(\sigma_+^*) - F(\sigma_-^*)] \cdot n_{\sigma_-,\tau} = \sum_{i=1}^{d} \sum_{\tau^\perp \| \hat{e}_i} [F(\sigma_-^* + \epsilon\, \hat{e}_i) - F(\sigma_-^*)] \cdot \epsilon^{d-1}\hat{e}_i$$

From the definition of derivative we have

$$F(\sigma_-^* + \epsilon\, \hat{e}_i) - F(\sigma_-^*) = \epsilon\, \frac{\partial F}{\partial x_i}(\sigma_-^*) + \epsilon\theta_\sigma,$$

where $\lim_{\epsilon \to 0} \theta_\sigma = 0$. It follows that

$$\begin{aligned}
\sum_{\substack{\tau \not\subseteq \partial P \\ \tau = \sigma_+ \cap \sigma_-}} [F(\sigma_+^*) - F(\sigma_-^*)] \cdot n_{\sigma_-,\tau} &= \sum_{i=1}^{d} \sum_{\tau^\perp \| \hat{e}_i} \left(\epsilon\, \frac{\partial F}{\partial x_i}(\sigma_-^*) + \epsilon\theta_\sigma \right) \cdot \epsilon^{d-1}\hat{e}_i \\
&= \sum_{i=1}^{d} \sum_{\tau^\perp \| \hat{e}_i} \left(\frac{\partial F}{\partial x_i}(\sigma_-^*) + \theta_\sigma \right) \epsilon^{d}\hat{e}_i \\
&= \sum_{\tau \not\subseteq \partial P} \left((\nabla \cdot F) + \theta_\sigma \right)\, \Delta V,
\end{aligned}$$

where $\Delta V = \epsilon^d$ is the volume of each d-cube in the original tiling $\mathcal{T}_\epsilon$. It follows that

$$\lim_{\epsilon \to 0} \sum_{\substack{\tau \not\subseteq \partial P \\ \tau = \sigma_+ \cap \sigma_-}} [F(\sigma_+^*) - F(\sigma_-^*)] \cdot n_{\sigma-,\tau} \;=\; \int_P (\nabla \cdot F)\; dV. \tag{6}$$

On combining (4), (5), and (6), we have

$$0 = \sum_{\substack{(\tau,\sigma) \\ \tau \subseteq \sigma \subseteq P}} F(\sigma^*) \cdot n_{\sigma,\tau} = \lim_{\epsilon \to 0} \sum_{\substack{(\tau,\sigma) \\ \tau \subseteq \sigma \subseteq P}} F(\sigma^*) \cdot n_{\sigma,\tau} = \int_{\partial P} F \cdot \hat{n}\; dS - \int_P (\nabla \cdot F)\; dV,$$

so that

$$\int_{\partial P} F \cdot \hat{n}\; dS = \int_P (\nabla \cdot F)\; dV,$$

∎

References

[BF48] T. Bonnesen and W. Fenchel. *Theorie der Konvexen Körper.* Chelsea, New York, USA, 1948.

[Jer96] D. Jerison. A Minkowski problem for electrostatic capacity. *Acta Math.*, 176:1–47, 1996.

[Lut93] E. Lutwak. The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem. *J. Differential Geometry*, 38:131–150, 1993.

[Sch93a] R. Schneider. *Convex Bodies: The Brunn-Minkowski Theory.* Cambridge University Press, New York, 1993.

[Sch93b] R. Schneider. Convex surfaces, curvature and surface area measures. In P. Gruber and J. M. Wills, editors, *Handbook of convex geometry*, pages 273–299. North-Holland, Amsterdam, 1993.