The Minkowski Problem for Polygons

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Abstract We describe a trivial solution to the Minkowski problem for polygons in the Euclidean plane.

Given a collection of unit normals u_i and corresponding edge lengths $\alpha_i > 0$, one might ask whether a convex polygon P exists whose boundary is described by the given normals and lengths. This question is known as the *Minkowski Problem.* While the Minkowski Problem (actually a theorem) is deep and non-trivial in higher dimensions (see, for example, [BF48, Sch93]), it is almost trivial in the planar case: By rotating the normals counter-clockwise by 90^o and lengthening each by its given edge length, we transform the normals into *actual edges.* Assuming we have listed the normals in counterclockwise order, we can just lay these oriented edges end-to-end and construct the polygonal closed curve that describes the desired polygon in a unique way up to translation; i.e. depending only on where we set down the pen to draw the first edge.

The details, which require some bookkeeping, are described as follows.

Theorem 1.1 (Minkowski Existence Theorem) *Suppose $u_1, u_2, \ldots, u_k \in \mathbb{R}^2$ are unit vectors that span $\mathbb{R}^2$, and suppose that $\alpha_1, \alpha_2, \ldots, \alpha_k > 0$. There exists a polygon $P \in \mathcal{P}^2$ having edge unit normals $u_1, u_2, \ldots, u_k$, and corresponding edge lengths $\alpha_1, \alpha_2, \ldots, \alpha_k$, if and only if*

$$\alpha_1 u_1 + \cdots + \alpha_k u_k = 0. \tag{1}$$

Moreover, such a polygon P is unique up to translation.

Proof: Suppose a polygon P has boundary data given by the normals u_i and edge-lengths α_i. Let ϕ denote the counter-clockwise rotation of $\mathbb{R}^2$ by the angle $\pi/2$. For each i let $v_i = \phi(\alpha_i u_i)$. Then each v_i is congruent to the i-th edge of the polygon P. Since the boundary of a convex polygon is a simple closed curve, we have

$$v_1 + \cdots + v_k = 0.$$

On applying ϕ^{-1} to this identity, we obtain (1).

Conversely, suppose that a family of unit vectors u_i and positive real numbers α_i satisfy (1), where the vectors u_i span $\mathbb{R}^2$. As above, let $v_i = \phi(\alpha_i u_i)$ for each i. Assume also that the vectors u_i (and therefore, the vectors v_i) are indexed in counterclockwise order around the circle. We will construct a polygon P having boundary data given by the normals u_i and edge-lengths α_i. The condition (1) implies that $v_1 + \cdots + v_k = 0$.

Denote

$$\begin{aligned} x_1 &= v_1 \\ x_2 &= v_1 + v_2 \\ &\vdots \\ x_k &= v_1 + \cdots + v_k = o \end{aligned}$$

and let P denote the convex hull of the points $x_1, x_2, \ldots, x_k$ (where $x_k = o$, the origin). We will show that each x_i is an extreme point of P. It will then follow that the x_i are the vertices of P, so that the edges of P are congruent to the vectors v_i, as required.

To show that each x_i is an extreme point, it is sufficient to consider the case of $x_k = o$. Moreover, since convex dependence relations are invariant under rigid motions, we may assume without loss of generality that v_1 points along the positive x-axis. Let $\jmath = (0,1)$. Note that $x_1 \cdot \jmath = v_1 \cdot \jmath = 0$, and that if $v_s \cdot \jmath < 0$ and $s \leq t \leq k$ then $v_t \cdot \jmath < 0$, since the v_i are ordered in counter-clockwise order. Moreover, since $\sum v_i = o$ and since the v_i span $\mathbb{R}^2$, we must have $v_2 \cdot \jmath > 0$.

Now suppose that o is *not* an extreme point of P. In this case, $o = a_1 x_1 + \cdots + a_k x_k$, where each $a_i > 0$ and $a_1 + \cdots + a_k = 1$. Note that

$$0 = o \cdot \jmath = \sum_i a_i (x_i \cdot \jmath). \tag{2}$$

Since $x_2 \cdot \jmath > 0$, it follows from (2) that some $x_s \cdot \jmath < 0$. Because $v_i \cdot \jmath < 0$ for all $i \geq s$, we have

$$0 = o \cdot \jmath = x_k \cdot \jmath = \left(x_s + \sum_{i>s} v_i \right) \cdot \jmath < 0,$$

a contradiction. It follows that o (and similarly each other x_i) must be an extreme point of P. It also follows that $x_s \cdot \jmath \geq 0$ for all s, so that v_1 (and similarly each other v_i) must be parallel to edges of P.

Since we have given an explicit reconstruction of the boundary of P from the normals u_i and edge lengths α_i, starting from a base point, in this case the origin o, it also follows that such a polygon P is unique up to choice of base point, that is, up to translation. ■

References

[BF48] T. Bonnesen and W. Fenchel. *Theorie der Konvexen Körper.* Chelsea, New York, USA, 1948.

[Sch93] R. Schneider. *Convex Bodies: The Brunn-Minkowski Theory.* Cambridge University Press, New York, 1993.