The Minkowski Problem for Simplices

Daniel A. Klain
Department of Mathematical Sciences
University of Massachusetts Lowell
Lowell, MA 01854 USA
Daniel_Klain@uml.edu

Abstract The Minkowski existence Theorem for polytopes follows from Cramer's Rule when attention is limited to the special case of simplices.

It is easy to see that a convex polygon in $\mathbb{R}^2$ is uniquely determined (up to translation) by the directions and lengths of its edges. This suggests the following (less easily answered) question in higher dimensions: given a collection of proposed facet normals and facet areas, is there a convex polytope in $\mathbb{R}^n$ whose facets fit the given data, and, if so, is the resulting polytope unique? This question (along with its answer) is known as the *Minkowski problem*.

For a polytope P in $\mathbb{R}^n$ denote by $V(P)$ the volume of P. If Q is a polytope in $\mathbb{R}^n$ having dimension strictly *less* than n, then denote $v(Q)$ the $(n-1)$-dimensional volume of Q. For any non-zero vector u, let P^u denote the face of P having u as an outward normal, and let P_u denote the orthogonal projection of P onto the hyperplane $u^\perp$.

The *Minkowski problem for polytopes* concerns the following specific question: Given a collection $u_1, \ldots, u_k$ of unit vectors and $\alpha_1, \ldots, \alpha_k > 0$, under what condition does there exist a polytope P having the u_i as its facet normals and the α_i as its facet areas; that is, such that $v(P^{u_i}) = \alpha_i$ for each i?

A necessary condition on the facet normals and facet areas is given by the following proposition [BF48, Sch93].

Proposition 1 *Suppose that a convex polytope $P \subseteq \mathbb{R}^n$ has facet normals $u_1, u_2, \ldots, u_k$ and corresponding facet areas $\alpha_1, \alpha_2, \ldots, \alpha_k$. Then*

$$\alpha_1 u_1 + \cdots + \alpha_k u_k = 0. \tag{1}$$

Proof: If $u \in \mathbb{R}^n$ is a unit vector, then $|u_i \cdot u|\alpha_i$ is equal to the area of the orthogonal projection of the i-th facet of P onto the hyperplane $u^\perp$. Summing over all facets whose outward normals form an acute angle with u we obtain

$$\sum_{u_i \cdot u > 0} (u_i \cdot u)\alpha_i = v(P_u),$$

where P_u denotes the orthogonal projection of P onto the hyperplane $u^\perp$. Summing analogously over all facets whose outward normals form an obtuse angle with u yields the value $-v(P_u)$. (In other words, P casts the *same* shadow onto the hyperplane $u^\perp$ from above as from below.)

Let $w = \alpha_1 u_1 + \cdots + \alpha_k u_k$. It now follows that

$$w \cdot u = \sum_i (u_i \cdot u)\alpha_i = \sum_{u_i \cdot u > 0} (u_i \cdot u)\alpha_i + \sum_{u_i \cdot u < 0} (u_i \cdot u)\alpha_i = v(P_u) - v(P_u) = 0.$$

In other words, $w \cdot u = 0$ for all u, so that $w = 0$. ■

Proposition 1 illustrates a necessary condition for the existence of a polytope having a given set of facet normals and facet areas. Minkowski discovered that the converse of Proposition 1 (along with some minor additional assumptions) is also true. In other words, the condition (1) is both necessary and (almost) sufficient, and moreover, determines a polytope that is unique up to translation. To be more precise, we have the following theorem.

Theorem 2 (Minkowski Existence Theorem) *Suppose $u_1, \ldots, u_k \in \mathbb{R}^n$ are unit vectors that span $\mathbb{R}^n$, and suppose that $\alpha_1, \ldots, \alpha_k > 0$. Then there exists a polytope $P \subseteq \mathbb{R}^n$, having facet unit normals $u_1, \ldots, u_k$ and corresponding facet areas $\alpha_1, \ldots, \alpha_k$, if and only if*

$$\alpha_1 u_1 + \cdots + \alpha_k u_k = 0.$$

Moreover, this polytope is unique up to translation.

For the classical proof of this theorem, see either of [BF48, Sch93]. Once the surface data are suitably defined, the Minkowski problem can also be generalized to the context of compact convex sets [Sch93] as well as to the p-mixed volumes of the Brunn-Minkowski-Firey theory [Lut93].

This note we addresses the following limited version of Minkowski's existence theorem.

Theorem 3 (Minkowski Existence Theorem for Simplices) *Suppose that $u_0, u_1, \ldots, u_n \in \mathbb{R}^n$ are unit vectors that span $\mathbb{R}^n$, and suppose that $\alpha_0, \alpha_1, \ldots, \alpha_n > 0$. Then there exists a simplex $S \subseteq \mathbb{R}^n$, having facet unit normals $u_0, u_1, \ldots, u_n$ and corresponding facet areas $\alpha_0, \alpha_1, \ldots, \alpha_n$, if and only if*

$$\alpha_0 u_0 + \alpha_1 u_1 + \cdots + \alpha_n u_n = 0.$$

Moreover, this simplex is unique up to translation.

Evidently Theorem 3 follows immediately from Theorem 2. Unfortunately the proof of the Minkowski Existence Theorem 2 is somewhat involved, while it is much easier to prove Theorem 3 *directly*, considering only the special case of simplices. Indeed, for simplices both existence and uniqueness follow more or less from Cramer's Rule.

Proof of Theorem 3: To begin, suppose that $S \subseteq \mathbb{R}^n$ is a simplex having facet unit normals $u_0, u_1, \ldots, u_n$ and corresponding facet areas $\alpha_0, \alpha_1, \ldots, \alpha_n$. It follows from the conditions on the u_i that S is non-degenerate, having positive volume.

Without loss of generality, suppose that the origin is a vertex of S, and denote by $x_1, \ldots, x_n$ the remaining vertices of S, arranged so that the vertex x_i lies opposite the ith facet. Let A denote the matrix whose columns are given by the vectors x_i, and suppose that the x_i are ordered so that A has positive determinant. In this instance $det(A) = n!\, V(S)$, where $V(S)$ denotes the volume of the simplex S. This follows from a combination of the base-height formula for the volume of a cone and induction on dimension.

Let $c(A)$ denote the cofactor matrix of A. Cramer's Rule asserts that

$$c(A)^T A = det(A) I, \tag{2}$$

where I is the $n \times n$ identity matrix. (See [Art91], for example, or any traditional linear algebra text.)

Let z_i denote the ith column of the matrix $c(A)$. The identity (2) asserts that $z_i \perp x_j$ for $j \neq i$. It follows that z_i is parallel to the facet normal u_i, and that $z_i = -|z_i| u_i$, since $z_i \cdot x_i = det(A) > 0$, while u_i points out of the simplex (away from the vertex x_i). Meanwhile, (2) also asserts that

$$z_i \cdot x_i = det(A) = n!\, V(S),$$

so that

$$-|z_i|(u_i \cdot x_i) = z_i \cdot x_i = n!\, V(S) = n!\, \frac{1}{n}\, \alpha_i(-u_i \cdot x_i),$$

where the final identity follows from the base-height formula for the volume of a cone, using the ith facet of S as the base. Hence, $|z_i| = \alpha_i (n-1)!$ and

$$z_i = -\alpha_i (n-1)!\, u_i.$$

In other words, the facet normals $u_1, \ldots, u_n$ and corresponding facet areas α_i are determined by the columns z_i of the cofactor matrix $c(A)$. The remaining facet normal u_0 and area α_0 is then determined by Minkowski's condition (1) in Proposition 1. This encoding of facet data into the cofactor matrix allows a simple proof of both existence and uniqueness for the simplex S given the data $\{u_i\}$ and $\{\alpha_i\}$.

To prove the uniqueness of S, note that $c(A) = det(A)A^{-T}$, by Cramer's Rule (2). It follows that $det(c(A)) = det(A)^{n-1}$ and that

$$A = det(A)c(A)^{-T} = det(c(A))^{\frac{1}{n-1}} c(A)^{-T}.$$

In other words, if two non-singular matrices A and B have the same cofactor matrix $c(A) = c(B)$, then $A = B$. It follows that if two simplices S and T each have the origin as a vertex and share the same facet normals and corresponding facet areas (for those facets incident to the origin), then S and T must have the same vertices, so that $S = T$.

More generally, if two simplices have the same facet normals and corresponding facet areas then they must be translates of one another.

Finally, to prove the *existence* of a simplex having the given facet data, suppose that $u_0, u_1, \ldots, u_n \in \mathbb{R}^n$ are unit vectors that span $\mathbb{R}^n$, that $\alpha_0, \alpha_1, \ldots, \alpha_n > 0$, and that

$$\alpha_0 u_0 + \alpha_1 u_1 + \cdots + \alpha_n u_n = 0.$$

Let C denote the matrix having columns $-\alpha_i(n-1)!\, u_i$ for $i > 0$. If A is the matrix having cofactor matrix C, then the columns of A, along with the origin, yield the vertices of a simplex having facet normals u_i and corresponding facet areas α_i. ■

Remark: The reader may observe that the preceding argument could be expressed more compactly in the language of Grassmann (alternating) tensors, thereby obscuring the role of Cramer's Rule in the proof.

References

[Art91] M. Artin. *Algebra*. Prentice-Hall, Englewood Cliffs, NJ, 1991.

[BF48] T. Bonnesen and W. Fenchel. *Theorie der Konvexen Körper*. Chelsea, New York, USA, 1948.

[Lut93] E. Lutwak. The Brunn-Minkowski-Firey theory. I. Mixed volumes and the Minkowski problem. *J. Differential Geometry*, 38:131–150, 1993.

[Sch93] R. Schneider. *Convex Bodies: The Brunn-Minkowski Theory*. Cambridge University Press, New York, 1993.