Linear optimization and the simplex method (with exercises)
by Dan Klain

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Corrections and comments are welcome.

1. Linear inequalities

Throughout this course we have considered systems of *linear equations* in one guise or another. Consider, for example, the system

$$\begin{aligned} x_1 + 3x_2 &= 18 \\ x_1 + x_2 &= 8 \\ 2x_1 + x_2 &= 14 \end{aligned} \tag{1}$$

in the two variables x_1, x_2. Using Gaussian elimination, or even just a little intuition, you can quickly determine that this system of equations is inconsistent; that is, it has no solutions. Equivalently, the three straight lines described by the system do not meet at a common point.

Suppose instead we consider the related system of *linear inequalities*:

$$\begin{gathered} x_1 + 3x_2 \leq 18 \\ x_1 + x_2 \leq 8 \\ 2x_1 + x_2 \leq 14 \end{gathered} \tag{2}$$

Instead of a straight line in $\mathbb{R}^2$, each of these inequalities determines a half-plane. The set of points (x_1, x_2) that satisfy this system are points that lie in all three half-planes, that is, the intersection of three half-planes. This set may be bounded or unbounded, shaped like a polygon or a polygonal cone. If we add two more conditions:

$$x_1 \geq 0 \qquad x_2 \geq 0$$

then the system (2) describes a pentagonal region on the first quadrant of $\mathbb{R}^2$, with one edge for each of the five inequalities:

$$\begin{gathered} x_1 + 3x_2 \leq 18 \\ x_1 + x_2 \leq 8 \\ 2x_1 + x_2 \leq 14 \\ x_1 \geq 0 \qquad x_2 \geq 0 \end{gathered} \tag{3}$$

Exercise: Try to draw this region! Can you list the coordinates of the five vertices (corners) of this pentagonal region? (Start by drawing the lines determined by each of the equations in (1) above.)

A typical problem in linear optimization runs as follows. Suppose a factory makes two kinds of candy. Each month the factory will produce x_1 cases of candy A and

x_2 cases of candy B. The factory uses three main ingredients for its candies: sugar, palm oil, and cocoa.

To make a case of candy A, the factory uses 100 units of sugar, 100 units of palm oil, and 200 units of cocoa. To make a case of candy B, the factory uses 300 units of sugar, 100 units of palm oil, and 100 units of cocoa. Suppose that the factory has on hand 1800 units of sugar, 800 units of palm oil, and 1400 units of cocoa. What possible combinations of candy A and candy B can be produced this month?

Since x_1 and x_2 count cases of candy, we may assume that $x_1, x_2 \geq 0$. Since we have only 1800 units of sugar available, we are subject to the *constraint*

$$100x_1 + 300x_2 \leq 1800$$

If instead we convert all of our measures to "hundreds of units," then

$$x_1 + 3x_2 \leq 18$$

More generally, the constraints governing all of our options for candy production are now given by the system of inequalities (3).

Now the question is, how much should we produce? For example, we could produce only candy A, and use up all of the cocoa to make 7 cases. But this would leave us with a lot of palm oil and sugar leftover. Or we could produce only candy B (how much at most?).

If you think about the possibilities, you will find no perfect answer; that is, for every choice of values for x_1 and x_2, there will be *some* raw materials leftover. This corresponds to the fact that the linear system (1) is inconsistent.

Suppose that market researchers tell us we can sell each case of candy A for \$5 and each case candy B for \$4 dollars. If we make x_1 cases of candy A and x_2 cases of candy B then our revenue $\hat{x}$ will be given by the function

$$\hat{x} = 5x_1 + 4x_2.$$

The question now is: how can we maximize revenue? More specifically, at which point of the polygon (3) is the function $5x_1 + 4x_2$ maximized?

The answer is hinted at by the following theorem, which we will not prove.

Theorem 1 (Extreme value theorem)**.** *If $\hat{x} = f(x_1, x_2, \ldots, x_n)$ is a linear function defined on a bounded polyhedron in $\mathbb{R}^n$, then $\hat{x}$ will attain its maximum (also minimum) value at some* **vertex** *of the polyhedron.*

This saves a lot of work. To find the maximum we could just list all the vertices, evaluate $5x_1 + 4x_2$ at each one, and pick the largest. For the candy problem above the constraint set has only 5 vertices, which we can find by drawing a picture. (You should do this as an exercise.)

Unfortunately this is not an acceptable method in practice. First of all, we cannot easily draw a picture if there are 3 variables instead of 2. And we cannot draw a picture at all if there are 4+ variables. Moreover, it's not even clear how we would find all the vertices. An authentic factory or allocation system might deal with hundreds of variables and a similar number of constraints, so we'll need to find a better way.

2. The simplex method (with equations)

The problem of the previous section can be summarized as follows.

> Maximize the function $\hat{x} = 5x_1 + 4x_2$
> subject to the constraints:
>
> $$\begin{aligned} x_1 + 3x_2 &\leq 18 \\ x_1 + x_2 &\leq 8 \\ 2x_1 + x_2 &\leq 14 \end{aligned}$$
>
> where we also assume that $x_1, x_2 \geq 0$.

Linear algebra provides powerful tools for simplifying linear *equations*. The first step in dealing with linear inequalities is to somehow transform them into equations, so that the technique of Gaussian elimination can be used.

For this purpose we introduce *slack variables*. Here is the idea. Instead of saying

$$x_1 + 3x_2 \leq 18$$

with $x_1, x_2 \geq 0$, we will say

$$x_1 + 3x_2 + x_3 = 18$$

with $x_1, x_2, x_3 \geq 0$. In other words, the new positive variable x_3 is "taking up the slack". Doing this for each constraint inequality enables us to transform the problem above into the following:

> Maximize the function $\hat{x} = 5x_1 + 4x_2$
> subject to the constraints:
>
> $$\begin{aligned} x_1 + 3x_2 + x_3 &= 18 \\ x_1 + x_2 + x_4 &= 8 \\ 2x_1 + x_2 + x_5 &= 14 \end{aligned} \tag{4}$$
>
> where we also assume that $x_1, x_2, x_3, x_4, x_5 \geq 0$.

Note that we have introduced three new *slack variables* x_3, x_4, x_5, one for each of the constraining inequalities in the original problem. The original variables x_1, x_2 are called the *basic variables*.

We summarize (4) as follows:

$$
\begin{array}{rrrrrcl}
5x_1 & +4x_2 & & & & = & \hat{x} \\
\hline
x_1 & +3x_2 & +x_3 & & & = & 18 \\
x_1 & +x_2 & & +x_4 & & = & 8 \\
2x_1 & +x_2 & & & +x_5 & = & 14
\end{array} \tag{5}
$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Begin by assuming that $x_1 = x_2 = 0$. This satisfies the equations (5) if we set $x_3 = 18, x_4 = 8$, and $x_5 = 14$. However, this is clearly not the answer to our optimization problem, since we are producing no candy at all! We have $\hat{x} = 5x_1 + 4x_2 = 0$ in this case. To maximize $\hat{x}$ we have to increase either x_1 or x_2 (or both). Let's try to increase x_1, being careful not to violate the rules imposed by (5).

> In the first equation we could set $x_1 = 18$ and $x_2 = x_3 = 0$.
> In the second equation we could set $x_1 = 8$ and $x_2 = x_4 = 0$.
> In the third equation we could set $x_1 = 7$ and $x_2 = x_5 = 0$.

Remember: We cannot set x_1 so large that any of the other variables become negative! This constrains how much x_1 can grow. Since *all* of the above conditions must be satisfied, we are forced to go with the *smallest* choice, and set $x_1 = 7$. To evaluate this case, we *pivot* on the variable x_1 in the last equation, using Gaussian elimination to get a new set of constraints:

$$
\begin{array}{rrrrrcl}
& \frac{3}{2}x_2 & & & -\frac{5}{2}x_5 & = & \hat{x} - 35 \\
\hline
& \frac{5}{2}x_2 & +x_3 & & -\frac{1}{2}x_5 & = & 11 \\
& \frac{1}{2}x_2 & & +x_4 & -\frac{1}{2}x_5 & = & 1 \\
x_1 & +\frac{1}{2}x_2 & & & +\frac{1}{2}x_5 & = & 7
\end{array} \tag{6}
$$

Now we can set $x_2 = x_5 = 0$ to obtain $x_1 = 7, x_3 = 11, x_4 = 1$ and $\hat{x} - 35 = 0$. In terms of our original problem, we have

$$(x_1, x_2) = (7, 0) \text{ and } \hat{x} = 35.$$

We have found a vertex $(7, 0)$ for the polygon at which $\hat{x} = 35$. In other words, our constraints will allow us to produce 7 hundred units of candy A and zero units of candy B giving us revenues of 35 hundred. Can we do better?

To determine this, check the objective equation, the top equation above the bar in (6). Since the coefficient of x_2 is positive (namely, $+\frac{3}{2}$), increasing x_2 will increase the whole sum.

The three equations below the bar in (6) constrain how large x_2 can become, since the remaining variable must never become negative.

In the first equation we could set $x_2 = \frac{22}{5}$ and $x_3 = x_5 = 0$.
In the second equation we could set $x_2 = 2$ and $x_4 = x_5 = 0$.
In the third equation we could set $x_2 = 14$ and $x_1 = x_5 = 0$.

The most stringent condition is the second, since $2 < \frac{22}{5} < 14$. So we pivot again, this time on the variable x_2 in the second constraint equation (beneath the bar). This pivoting now results in a new set of equations:

$$
\begin{array}{cccccccc}
 & & & -3x_4 & -x_5 & = & \hat{x} - 38 \\
\hline
 & & x_3 & -5x_4 & +2x_5 & = & 6 \\
 & x_2 & & +2x_4 & -x_5 & = & 2 \\
x_1 & & & -x_4 & +x_5 & = & 6
\end{array} \tag{7}
$$

Setting $x_4 = x_5 = 0$ yields $x_1 = 6, x_2 = 2, x_3 = 11$ and $\hat{x} - 38 = 0$. In terms of our original problem, we have

$$(x_1, x_2) = (6, 2), \text{ and } \hat{x} = 38. \tag{8}$$

The point $(6, 2)$ is a vertex for the constraint polygon at which $\hat{x} = 38$. In other words, our constraints will allow us to produce 6 hundred units of candy A and 2 units of candy B giving us revenues of 38 hundred.

Can we do still better? No! Notice in the objective function on top of the bar in (7) all the coefficients on the left side are either zero or negative:

$$-3x_4 - x_5 = \hat{x} - 38 \qquad \Longrightarrow \qquad \hat{x} = 38 - 3x_4 - x_5$$

If we increase either x_4 or x_5 then $\hat{x}$ will only decrease. Therefore, the maximum point for our revenue function $\hat{x}$ is given by (8).

3. The simplex method (with tableaux)

The discussion of the previous section is cluttered with many variables. When solving linear equations, it is customary to drop the variables and perform Gaussian elimination on a matrix of coefficients. The technique used in the previous section to maximize the function $\hat{x}$, called the *simplex method*, is also typically performed on a matrix of coefficients, usually referred to (in this context) as a *tableau*. The sequence of tableaux we used to solve the candy factory problem are the following:

$$
\begin{array}{cc|ccc|c}
5 & 4 & 0 & 0 & 0 & \hat{x} \\
\hline
1 & 3 & 1 & 0 & 0 & 18 \\
1 & 1 & 0 & 1 & 0 & 8 \\
2 & 1 & 0 & 0 & 1 & 14
\end{array}
\Longrightarrow
\begin{array}{cc|ccc|c}
0 & \frac{3}{2} & 0 & 0 & -\frac{5}{2} & -35 + \hat{x} \\
\hline
0 & \frac{5}{2} & 1 & 0 & -\frac{1}{2} & 11 \\
0 & \frac{1}{2} & 0 & 1 & -\frac{1}{2} & 1 \\
1 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 7
\end{array}
\Longrightarrow
\begin{array}{cc|ccc|c}
0 & 0 & 0 & -3 & -1 & -38 + \hat{x} \\
\hline
0 & 0 & 1 & -5 & 2 & 6 \\
0 & 1 & 0 & 2 & -1 & 2 \\
1 & 0 & 0 & -1 & 1 & 6
\end{array}
$$

In each step the circled position is the pivot for the next step.

We will now do an example using only tableaux.

Maximize the function $\hat{x} = 2x_1 + x_2$
subject to the constraints:

$$\begin{aligned} x_1 + x_2 &\leq 40 \\ 3x_1 + x_2 &\leq 90 \\ x_1 + 2x_2 &\leq 60 \\ x_1, x_2 &\geq 0. \end{aligned}$$

Solution: The initial tableau is

variables	x_1	x_2	x_3	x_4	x_5	
objective function	2	1	0	0	0	$\hat{x}$
	1	1	1	0	0	40
constraints	3	1	0	1	0	90
	1	2	0	0	1	60

This tableau corresponds to setting the basic variables $x_1 = x_2 = 0$ and slack variables $x_3 = 40$, $x_4 = 90$, and $x_5 = 60$. At this point $\hat{x} = 0$.

The positive values 2 and 1 in the top row tell us we can either try to increase x_1 or x_2. Let's increase x_1. The three rows beneath the second bar give the constraints.

First row: $x_1 \leq 40$.
Second row: $x_1 \leq \frac{90}{3} = 30$.
Third row: $x_1 \leq 60$.

All of the constraints must be satisfied, so we set $x_1 = 30$ by pivoting at the second constraint row, first column, to obtain

$$\begin{array}{cc|ccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \hline 2 & 1 & 0 & 0 & 0 & \hat{x} \\ \hline 1 & 1 & 1 & 0 & 0 & 40 \\ 3 & 1 & 0 & 1 & 0 & 90 \\ 1 & 2 & 0 & 0 & 1 & 60 \end{array} \Longrightarrow \begin{array}{cc|ccc|r} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \hline 0 & \frac{1}{3} & 0 & -\frac{2}{3} & 0 & -60 + \hat{x} \\ \hline 0 & \frac{2}{3} & 1 & -\frac{1}{3} & 0 & 10 \\ 1 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 30 \\ 0 & \frac{5}{3} & 0 & -\frac{1}{3} & 1 & 30 \end{array}$$

We still have a positive value $\frac{1}{3}$ in the top row, which means we can increase x_2 as well. Once again, the three rows beneath the second bar give the constraints.

First row: $x_2 \leq \frac{3}{2}(10) = 15$.
Second row: $x_2 \leq 3(30) = 90$.
Third row: $x_2 \leq \frac{3}{5}(30) = 18$.

The most strict of the conditions is the first, so we pivot at the circled position to obtain:

$$\begin{array}{cc|ccc|r}
x_1 & x_2 & x_3 & x_4 & x_5 & \\
\hline
0 & \frac{1}{3} & 0 & -\frac{2}{3} & 0 & -60+\hat{x} \\
\hline
0 & \frac{2}{3} & 1 & -\frac{1}{3} & 0 & 10 \\
1 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 30 \\
0 & \frac{5}{3} & 0 & -\frac{1}{3} & 1 & 30
\end{array}
\quad \Longrightarrow \quad
\begin{array}{cc|ccc|r}
x_1 & x_2 & x_3 & x_4 & x_5 & \\
\hline
0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & -65+\hat{x} \\
\hline
0 & 1 & \frac{3}{2} & -\frac{1}{2} & 0 & 15 \\
1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 25 \\
0 & 0 & -\frac{5}{2} & \frac{1}{2} & 1 & 5
\end{array}$$

Since all of the top row values are negative, we are finished. The value of $\hat{x}$ is maximized at the vertex $(x_1, x_2) = (25, 15)$, where $\hat{x} = 65$.

Exercises:

1. Use the simplex method to maximize the function $\hat{x} = 5x_1 + 2x_2$ subject to the constraints:

$$\begin{array}{rcl}
4x_1 + 2x_2 & \leq & 32 \\
2x_1 + x_2 & \leq & 18 \\
x_1 + x_2 & \leq & 12 \\
x_1 + 4x_2 & \leq & 36 \\
x_1, x_2 & \geq & 0.
\end{array}$$

2. Use the simplex method to maximize the function $\hat{x} = 3x_1 + 2x_2$ subject to the constraints:

$$\begin{array}{rcl}
x_1 + 2x_2 & \leq & 24 \\
x_1 + x_2 & \leq & 21 \\
-x_1 + x_2 & \leq & 9 \\
x_1, x_2 & \geq & 0.
\end{array}$$

3. Sketch the constraint polygon from problem **1.** above.

4. Sketch the constraint polygon from problem **2.** above. Which vertices did you visit while running the simplex method in problem **2.**?

5. Use the simplex method to maximize the function $\hat{x} = 2x_1 + x_2 + x_3$ subject to the constraints:

$$\begin{aligned} x_1 + x_2 + x_3 &\leq 40 \\ 2x_1 + x_3 &\leq 30 \\ x_1 + x_2 + 3x_3 &\leq 60 \\ x_1 + 4x_2 + x_3 &\leq 120 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

6. Use the simplex method to maximize the function

$$\hat{x} = x_1 + 5x_2 + x_3$$

subject to the constraints:

$$\begin{aligned} x_1 + 3x_2 + x_3 &\leq 4 \\ 2x_2 + x_3 &\leq 2 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

7. Use the simplex method to maximize the function $\hat{x} = 3x_1 + 2x_2 + x_3$ subject to the constraints:

$$\begin{aligned} x_1 &\leq 10 \\ x_2 &\leq 10 \\ x_3 &\leq 10 \\ x_1 + x_2 + x_3 &\leq 15 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

8. Sketch the constraint polyhedron from problem **6.** above. Which vertices did you visit while running the simplex method in problem **6.**?

9. Howard wants to increase his daily intake of protein. He has foolishly decided to eat only steak, chicken, and fish. Each serving of steak has 15g of protein and costs \$8. Each serving of chicken has 10g of protein and costs \$5. Each serving of fish has 20g of protein and costs \$4. Howard has a daily budget of \$40. Moreover, he likes variety and will not eat more than 4 servings of fish, nor will he eat more than 7 servings of chicken and steak combined. The problem is to find a diet acceptable to Howard that maximizes his daily protein intake.

Describe this problem as a linear optimization problem, and set up the inital tableau for applying the simplex method. (But do not solve – unless you really want to, in which case it's ok to have partial (fractional) servings.)

Answers to Selected Exercises:

1.

$$
\begin{array}{cc|cccc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\
\hline
5 & 2 & 0 & 0 & 0 & 0 & \hat{x} \\
\hline
4 & 2 & 1 & 0 & 0 & 0 & 32 \\
2 & 1 & 0 & 1 & 0 & 0 & 18 \\
1 & 1 & 0 & 0 & 1 & 0 & 12 \\
1 & 4 & 0 & 0 & 0 & 1 & 36
\end{array}
\Longrightarrow
\begin{array}{cc|cccc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\
\hline
0 & -\frac{1}{2} & -\frac{5}{4} & 0 & 0 & 0 & \hat{x}-40 \\
\hline
1 & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 8 \\
0 & 0 & -\frac{1}{2} & 1 & 0 & 0 & 2 \\
0 & \frac{1}{2} & -\frac{1}{4} & 0 & 1 & 0 & 4 \\
0 & \frac{7}{2} & -\frac{1}{4} & 0 & 0 & 1 & 28
\end{array}
$$

so that $\hat{x} = 5x_1 + 2x_2$ is maximized at $(x_1, x_2) = (8, 0)$ and $\hat{x} = 40$.

2. *First solution*

$$
\begin{array}{cc|ccc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & \\
\hline
3 & 2 & 0 & 0 & 0 & \hat{x} \\
\hline
1 & 2 & 1 & 0 & 0 & 24 \\
1 & 1 & 0 & 1 & 0 & 21 \\
-1 & 1 & 0 & 0 & 1 & 9
\end{array}
\Longrightarrow
\begin{array}{cc|ccc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & \\
\hline
0 & -1 & 0 & -1 & 0 & \hat{x}-63 \\
\hline
0 & 1 & 1 & -1 & 0 & 3 \\
1 & 1 & 0 & 1 & 0 & 21 \\
0 & 2 & 0 & 1 & 1 & 30
\end{array}
$$

so that $\hat{x} = 3x_1 + 2x_2$ is maximized at $(x_1, x_2) = (21, 0)$ and $\hat{x} = 63$.

2. *Alternate solution*

$$
\begin{array}{cc|ccc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & \\
\hline
3 & 2 & 0 & 0 & 0 & \hat{x} \\
\hline
1 & 2 & 1 & 0 & 0 & 24 \\
1 & 1 & 0 & 1 & 0 & 21 \\
-1 & 1 & 0 & 0 & 1 & 9
\end{array}
\Longrightarrow
\begin{array}{cc|ccc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & \\
\hline
5 & 0 & 0 & 0 & -2 & \hat{x}-18 \\
\hline
3 & 0 & 1 & 0 & -2 & 6 \\
2 & 0 & 0 & 1 & -1 & 12 \\
-1 & 1 & 0 & 0 & 1 & 9
\end{array}
\Longrightarrow \cdots
$$

$$
\begin{array}{cc|ccc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & \\
\hline
0 & 0 & -\frac{5}{3} & 0 & \frac{4}{3} & \hat{x}-28 \\
\hline
1 & 0 & \frac{1}{3} & 0 & -\frac{2}{3} & 2 \\
0 & 0 & -\frac{2}{3} & 1 & \frac{1}{3} & 8 \\
0 & 1 & \frac{1}{3} & 0 & \frac{1}{3} & 11
\end{array}
\Longrightarrow
\begin{array}{cc|ccc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & \\
\hline
0 & 0 & 1 & -4 & 0 & \hat{x}-60 \\
\hline
1 & 0 & -1 & 2 & 0 & 18 \\
0 & 0 & -2 & 3 & 1 & 24 \\
0 & 1 & 1 & -1 & 0 & 3
\end{array}
\Longrightarrow \cdots
$$

$$
\cdots \Longrightarrow
\begin{array}{cc|ccc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & \\
\hline
0 & -1 & 0 & -3 & 0 & \hat{x}-63 \\
\hline
1 & 0 & 0 & 1 & 0 & 21 \\
0 & 0 & 0 & 1 & 1 & 30 \\
0 & 1 & 1 & -1 & 0 & 3
\end{array}
$$

so that $\hat{x} = 3x_1 + 2x_2$ is maximized at $(x_1, x_2) = (21, 0)$ and $\hat{x} = 63$.

6. *First solution*

$$
\begin{array}{ccc|cc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & \\
\hline
1 & 5 & 1 & 0 & 0 & \hat{x} \\
\hline
1 & 3 & 1 & 1 & 0 & 4 \\
0 & 2 & 1 & 0 & 1 & 2
\end{array}
\implies
\begin{array}{ccc|cc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & \\
\hline
0 & 2 & 0 & -1 & 0 & \hat{x}-4 \\
\hline
1 & 3 & 1 & 1 & 0 & 4 \\
0 & 2 & 1 & 0 & 1 & 2
\end{array}
\implies \cdots
$$

$$
\begin{array}{ccc|cc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & \\
\hline
0 & 0 & -1 & -1 & -1 & \hat{x}-6 \\
\hline
1 & 0 & -\frac{1}{2} & 1 & -\frac{3}{2} & 1 \\
0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 1
\end{array}
\quad \text{so} \quad \boxed{x_1 = 1, x_2 = 1, x_3 = 0, \text{ and } \hat{x} = 6}
$$

6. *Alternate solution*

$$
\begin{array}{ccc|cc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & \\
\hline
1 & 5 & 1 & 0 & 0 & \hat{x} \\
\hline
1 & 3 & 1 & 1 & 0 & 4 \\
0 & 2 & 1 & 0 & 1 & 2
\end{array}
\implies
\begin{array}{ccc|cc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & \\
\hline
1 & 0 & -\frac{3}{2} & 0 & -\frac{5}{2} & \hat{x}-5 \\
\hline
1 & 0 & -\frac{1}{2} & 1 & -\frac{3}{2} & 1 \\
0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 1
\end{array}
\implies \cdots
$$

$$
\begin{array}{ccc|cc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & \\
\hline
0 & 0 & -1 & -1 & -1 & \hat{x}-6 \\
\hline
1 & 0 & -\frac{1}{2} & 1 & -\frac{3}{2} & 1 \\
0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 1
\end{array}
\quad \text{so} \quad \boxed{x_1 = 1, x_2 = 1, x_3 = 0, \text{ and } \hat{x} = 6}
$$

9. Let s = servings of steak, c = servings of chicken, and f = servings of fish. Then we must maximize the function $\hat{x} = 15s + 10c + 20f$,
subject to the constraints:

$$
\begin{aligned}
8s + 5c + 4f &\leq 40 \\
f &\leq 4 \\
s + c &\leq 7 \\
\\
x_1, x_2, x_3 &\geq 0.
\end{aligned}
$$

Initial tableau is:

$$
\begin{array}{ccc|ccc|c}
s & c & f & x_4 & x_5 & x_6 & \\
\hline
15 & 10 & 20 & 0 & 0 & 0 & \hat{x} \\
\hline
8 & 5 & 4 & 1 & 0 & 0 & 40 \\
0 & 0 & 1 & 0 & 1 & 0 & 4 \\
1 & 1 & 0 & 0 & 0 & 1 & 7
\end{array}
$$

4. Standard Form

So far we have considered problems of the form:

Maximize the function $\hat{x} = a_1 x_1 + \cdots + a_n x_n$
subject to the constraints:

$$\begin{aligned} c_{11}x_1 + \cdots + c_{1n}x_n &\leq b_1 \\ &\vdots \\ c_{m1}x_1 + \cdots + c_{mn}x_n &\leq b_m \end{aligned}$$

where $x_1, \cdots, x_n \geq 0$.

In this type of problem we are *maximizing* the objective function $\hat{x}$, using *non-negative* variables $x_i \geq 0$ and constraints of the form

$$c_{i1}x_1 + \cdots + c_{in}x_n \leq b_i,$$

where the linear function is *bounded above* by a constant b_i.

Problems stated in this manner are said to be in *standard form*.

❧

What if a linear programming problem is not given in this form? In most case some easy adjustments can be made to re-state the problem in standard form, after which the simplex method may be used to solve it.

Here are some tips for making such adjustments:

- If the problem asks you to *minimize* $\hat{x}$, this can be accomplished by instead maximizing the function $-\hat{x}$ and taking the negative of the resulting answer. For example, the problem, "Minimize $x - y$," can be solved by *maximizing* $y - x$ and then negating the final result.
- Similarly, if a constraint equation is given with the inequality in the wrong direction ($\geq$), then multiplication of both sides by -1 will reverse it. For example,

  $$x + y \geq 7$$

  can be replaced with the equivalent

  $$-x - y \leq -7$$

- If a constraint equation is given as an identity (=), it can be replaced with two inequalities having this identity as intersection. For example,

$$x + y = 7$$

can be replaced with the equivalent pair

$$x + y \leq 7$$

$$x + y \geq 7$$

or, even better,

$$x + y \leq 7$$
$$-x - y \leq -7$$

- If a variable x_i is bounded by a number less than zero, a substitution can be made:

$$x_1 \geq -4$$

becomes

$$u_1 \geq 0 \quad \text{where } u_1 = x_1 + 4$$

- If a variable x_i is unbounded, a substitution with two new variables can be made:

$$x_1 = u_1 - v_1 \quad \text{where } u_1, v_1 \geq 0$$

❧

Here are some examples of putting linear programs into standard form.

The linear program:

Minimize the function $\hat{x} = x_1 - 3x_2 + x_3$
subject to the constraints:

$$\begin{aligned} 2x_1 + x_2 + x_3 &\leq 20 \\ x_1 + 3x_2 - x_3 &\geq 10 \end{aligned}$$

where $x_1, x_2, x_3 \geq 0$.

is put into standard form as follows:

Maximize the function $\hat{y} = -x_1 + 3x_2 - x_3$
subject to the constraints:

$$\begin{aligned} 2x_1 + x_2 + x_3 &\leq 20 \\ -x_1 - 3x_2 + x_3 &\leq -10 \end{aligned}$$

where $x_1, x_2, x_3 \geq 0$.

Here we turned a minimization problem into an equivalent maximization problem, and we also negated a constraint inequality so that all constraints are bounded above ($\leq$). After solving this maximization problem a solution to the original occurs at the same values of x_1, x_2, x_3 by setting $\hat{x} = -\hat{y}$.

❧

In another example, we can put the linear program:

Maximize the function $\hat{x} = x_1 + x_2$
subject to the constraints:

$$\begin{aligned} 2x_1 + x_2 &\leq 14 \\ x_1 + 3x_2 &\leq 10 \end{aligned}$$

where $x_1 \geq 0$.

into standard form as follows:

Maximize the function $\hat{x} = x_1 + u_1 - u_2$
subject to the constraints:

$$\begin{aligned} 2x_1 + u_1 - u_2 &\leq 14 \\ x_1 + 3u_1 - 3u_2 &\leq 10 \end{aligned}$$

where $x_1, u_1, u_2 \geq 0$.

Here we made the substitution $x_2 = u_1 - u_2$ so that the unbounded variable x_2 is now a difference of non-negative variables u_1, u_2.

❧

Following the list of adjustments above we can turn any linear program into an equivalent program in standard form. At this point we would like to be able to use the simplex method to solve. Unfortunately, these adjustments can introduce a problem. Consider the first example above:

Maximize the function $\hat{y} = -x_1 + 3x_2 - x_3$
subject to the constraints:

$$\begin{aligned} 2x_1 + x_2 + x_3 &\leq 20 \\ -x_1 - 3x_2 + x_3 &\leq -10 \end{aligned}$$

where $x_1, x_2, x_3 \geq 0$.

This problem might not have any solutions at all. Unlike the problems in Section 1 above, we cannot set all $x_i = 0$ for an initial feasible guess, because $0 \not\leq -10$. The presence of *negative values* as upper bounds (such as -10 in the display above) could make some linear programs *infeasible*, meaning that there are no values of x_i that satisfy the system. Geometrically this would imply that the polyhedron is actually empty.

We will need to address two questions. First, if there are negative upper bounds, how do we determine if a linear program has any solutions? Second, how can we adjust the system to eliminate those negative upper bounds and then use the simplex method to solve? These questions will be answered in the next section.

Exercises:

1. Put the following linear program into standard form:

Maximize the function $\hat{x} = x_1 + 6x_2 + x_3$
subject to the constraints:

$$\begin{aligned} x_1 + x_3 &\geq -9 \\ x_1 - 3x_2 + x_3 &\leq 13 \end{aligned}$$

where $x_1, x_2 \geq 0$ and $x_3 \leq 0$.

2. Put the following linear program into standard form:

Minimize the function $\hat{x} = x_1 + x_2 + x_3$
subject to the constraints:

$$\begin{aligned} 8x_1 - x_2 &\leq 20 \\ -x_1 + 3x_2 + 4x_3 &\leq 13 \end{aligned}$$

where $x_1, x_2 \geq 0$ and $x_3 \geq 4$.

3. Put the following linear program into standard form:

Maximize the function $\hat{x} = x_1 + x_2$
subject to the constraints:

$$\begin{aligned} 4x_1 + 2x_2 &\leq 21 \\ 2x_1 + x_2 &\leq 24 \\ 3x_1 + 7x_2 &\leq 20 \end{aligned}$$

where $x_1 \geq 0$.

4. Put the following linear program into standard form:

Maximize the function $\hat{x} = x_1 + x_2$
subject to the constraints:

$$\begin{aligned} 4x_1 - 2x_2 &\leq 21 \\ 2x_1 + x_2 &\leq 24 \\ 3x_1 - 7x_2 &\leq 20 \end{aligned}$$

where $x_1 \geq 0$ and $x_2 \leq 0$.

5. Put the following linear program into standard form:

Maximize the function $\hat{x} = 3x_1 - 2x_2 + x_3$
subject to the constraints:

$$\begin{aligned} x_1 - x_2 + x_3 &\leq 15 \\ x_1 - 2x_2 + x_3 &\leq 12 \end{aligned}$$

where $0 \leq x_1 \leq 10$, $-10 \leq x_2 \leq 0$, and $0 \leq x_3 \leq 10$.

❧

Answers to Selected Exercises:

1. Set $u_3 = -x_3$, and multiply second contraint by -1:

Maximize the function $\hat{x} = x_1 + 6x_2 - u_3$
subject to the constraints:

$$\begin{aligned} -x_1 + u_3 &\leq 9 \\ x_1 - 3x_2 - u_3 &\leq 13 \end{aligned}$$

where $x_1, x_2, u_3 \geq 0$.

2. Maximize the negative of the given objective, and also substitute $u_3 = x_3 - 4$ and simplify:

Maximize the function $\hat{y} = -x_1 - x_2 - x_3$
subject to the constraints:

$$\begin{aligned} 8x_1 - x_2 &\leq 20 \\ -x_1 + 3x_2 + 4u_3 &\leq -3 \end{aligned}$$

where $x_1, x_2, u_3 \geq 0$.

3. Since x_2 is uncontrained, set $x_2 = u_2 - v_2$, where $u_2, v_2 \geq 0$:

Maximize the function $\hat{x} = x_1 + u_2 - v_2$
subject to the constraints:

$$\begin{aligned} 4x_1 + 2u_2 - 2v_2 &\leq 21 \\ 2x_1 + u_2 - v_2 &\leq 24 \\ 3x_1 + 7u_2 - 7v_2 &\leq 20 \end{aligned}$$

where $x_1, u_2, v_2 \geq 0$.

4. Set $u_2 = -x_2$:

Maximize the function $\hat{x} = x_1 - u_2$
subject to the constraints:

$$\begin{aligned} 4x_1 + 2u_2 &\leq 21 \\ 2x_1 - u_2 &\leq 24 \\ 3x_1 + 7u_2 &\leq 20 \end{aligned}$$

where $x_1, u_2 \geq 0$.

5. Set $u_2 = -x_2$ and move the upper bounds into the table of constraints:

Maximize the function $\hat{x} = 3x_1 - 2x_2 + x_3$
subject to the constraints:

$$\begin{aligned} x_1 + u_2 + x_3 &\leq 15 \\ x_1 + 2u_2 + x_3 &\leq 12 \\ x_1 &\leq 10 \\ u_2 &\leq 10 \\ x_3 &\leq 10 \end{aligned}$$

where $x_1, u_2, x_3 \geq 0$.

5. Feasible points

Suppose we want to solve the problem:

Maximize the function $\hat{x} = x_1 + x_2$
subject to the constraints:

$$\begin{aligned} x_1 + x_2 &\geq 1 \\ -2x_1 + x_2 &\geq -6 \\ x_1 + 3x_2 &\leq 10 \\ -2x_1 + x_2 &\leq 1 \end{aligned}$$

where $x_1, x_2 \geq 0$.

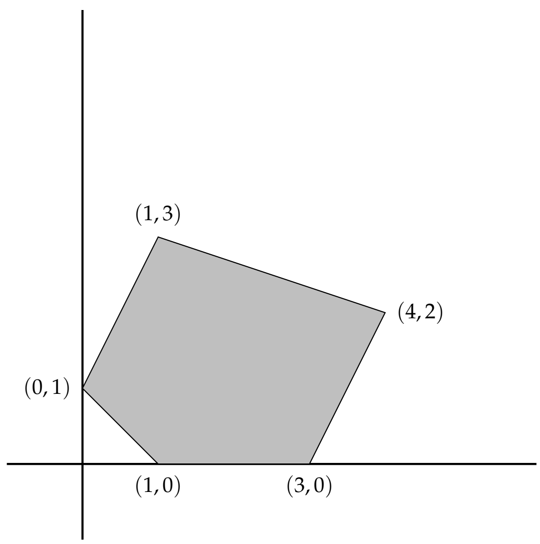

Figure 1. The feasible region described by (9).

In standard form, this becomes:

Maximize the function $\hat{x} = x_1 + x_2$
subject to the constraints:

$$\begin{aligned} -x_1 - x_2 &\leq -1 \\ 2x_1 - x_2 &\leq 6 \\ x_1 + 3x_2 &\leq 10 \\ -2x_1 + x_2 &\leq 1 \end{aligned} \tag{9}$$

where $x_1, x_2 \geq 0$.

Immediately there is a difficulty. The inequality $-x_1 - x_2 \leq -1$ is not satisfied when $x_1 = x_2 = 0$, so it's not clear for the moment whether ***any*** points (x_1, x_2) exist that satisfy this problem. Before we can find an optimal (maximizing) point, we need to know if there are any points in this object at all. Perhaps the list (9) of inequalities describes the empty set.

The space of points described by the constraint inequalities (9) is called the set of *feasible points*. In the examples of Section 1 the origin (all $x_i = 0$) was an obvious feasible point, but in this example we need to determine if any feasible points exist before we can maximize any objective function.

One way to address this concern is to sketch the region described by (9). In this particular example the result would be the region in Figure 1. However, we have seen that two-dimensional drawings are not always possible for common applications, so we need an algebraic approach. To address this we temporarily discard the objective function $\hat{x} = x_1 + x_2$ and replace it with a new one: $\hat{x} = -z$. We also adjust the constraints to write an *auxiliary* problem:

Maximize the function $\hat{x} = -z$
subject to the constraints:

$$\begin{aligned} -x_1 - x_2 - z &\leq -1 \\ 2x_1 - x_2 - z &\leq 6 \\ x_1 + 3x_2 - z &\leq 10 \\ -2x_1 + x_2 - z &\leq 1 \end{aligned} \tag{10}$$

where $x_1, x_2, z \geq 0$.

Notice that this new problem *is* feasible, since $(x_1, x_2, z) = (0, 0, 1)$ is obviously a feasible point of (10). If we can maximize $\hat{x} = -z$ at the value $z = 0$, then we have found a point $(x_1, x_2, 0)$ that lies in the feasible region of the original problem. If the maximum of $\hat{x}$ is strictly negative, then the original problem was *infeasible* and unsolvable, since the constraints describe an empty set of points.

The strategy for solving (10) involves two parts: first, a positivity adjustment, after which we can follow the procedures from earlier. In the initial tableau:

x_1	x_2	z	x_3	x_4	x_5	x_6	
0	0	−1	0	0	0	0	$\hat{x}$
−1	−1	−1	1	0	0	0	−1
2	−1	−1	0	1	0	0	6
1	3	−1	0	0	1	0	10
−2	1	−1	0	0	0	0	1

To begin, pivot in the column of the auxiliary objective z, placing the pivot in the same row as the **lowest negative** value appearing in the *last* column:

$$\begin{array}{rr|r|rrrr|r}
x_1 & x_2 & z & x_3 & x_4 & x_5 & x_6 & \\
\hline
0 & 0 & -1 & 0 & 0 & 0 & 0 & \hat{x} \\
\hline
-1 & -1 & -1 & 1 & 0 & 0 & 0 & -1 \\
2 & -1 & -1 & 0 & 1 & 0 & 0 & 6 \\
1 & 3 & -1 & 0 & 0 & 1 & 0 & 10 \\
-2 & 1 & -1 & 0 & 0 & 0 & 1 & 1
\end{array}$$

After this step the entire last column will be positive, and we can continue pivoting as usual:

$$\begin{array}{rr|r|rrrr|r}
x_1 & x_2 & z & x_3 & x_4 & x_5 & x_6 & \\
\hline
1 & 1 & 0 & -1 & 0 & 0 & 0 & \hat{x}+1 \\
\hline
1 & 1 & 1 & -1 & 0 & 0 & 0 & 1 \\
3 & 0 & 0 & -1 & 1 & 0 & 0 & 7 \\
2 & 4 & 0 & -1 & 0 & 1 & 0 & 11 \\
-1 & 2 & 0 & -1 & 0 & 0 & 1 & 2
\end{array}$$

to obtain

$$\begin{array}{rr|r|rrrr|r}
x_1 & x_2 & z & x_3 & x_4 & x_5 & x_6 & \\
\hline
0 & 0 & -1 & 0 & 0 & 0 & 0 & \hat{x} \\
\hline
1 & 1 & 1 & -1 & 0 & 0 & 0 & 1 \\
0 & -3 & -3 & 2 & 1 & 0 & 0 & 4 \\
0 & 2 & -2 & 1 & 0 & 1 & 0 & 9 \\
0 & 3 & 1 & -2 & 0 & 0 & 1 & 3
\end{array}$$

This completes the auxiliary problem. **Since $\hat{x}$ is maximized at** 0**,** we have found a feasible point for the original system at $(x_1, x_2) = (1, 0)$.

To solve the original maximization problem, remove the z-column and replace the objective function with the original $\hat{x} = x_1 + x_2$:

$$\begin{array}{rr|rrrr|r}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\
\hline
1 & 1 & 0 & 0 & 0 & 0 & \hat{x} \\
\hline
1 & 1 & -1 & 0 & 0 & 0 & 1 \\
0 & -3 & 2 & 1 & 0 & 0 & 4 \\
0 & 2 & 1 & 0 & 1 & 0 & 9 \\
0 & 3 & -2 & 0 & 0 & 1 & 3
\end{array} \tag{11}$$

and solve for maximal $\hat{x}$ as before.

Exercises:

1. Maximize $x_1 + 2x_2 + 3x_3$ subject to the constraints:

$$\begin{aligned} x_1 + x_2 + x_3 &\geq 10 \\ x_1 + x_2 + x_3 &\leq 10 \\ x_1 - x_2 &\leq 2 \\ x_2 + 2x_3 &\leq 4 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

2. Finish solving the maximization problem from the example in this section, continuing with the tableau (11).