Orthogonal Projections and Reflections (with exercises)
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Corrections and comments are welcome.

Orthogonal Projections

Let $X_1, \ldots, X_k$ be a family of linearly independent (column) vectors in $\mathbb{R}^n$, and let

$$W = \mathrm{Span}(X_1, \ldots, X_k).$$

In other words, the vectors $X_1, \ldots, X_k$ form a basis for the k-dimensional subspace W of $\mathbb{R}^n$.

Suppose we are given another vector $Y \in \mathbb{R}^n$. How can we project Y onto W orthogonally? In other words, can we find a vector $\hat{Y} \in W$ so that $Y - \hat{Y}$ is orthogonal (perpendicular) to all of W? See Figure 1.

To begin, translate this question into the language of matrices and dot products. We need to find a vector $\hat{Y} \in W$ such that

$$(Y - \hat{Y}) \perp Z, \text{ for all vectors } Z \in W. \tag{1}$$

Actually, it's enough to know that $Y - \hat{Y}$ is perpendicular to the vectors $X_1, \ldots, X_k$ that span W. This would imply that (1) holds. (Why?)

Expressing this using dot products, we need to find $\hat{Y} \in W$ so that

$$X_i^T (Y - \hat{Y}) = 0, \text{ for all } i = 1, 2, \ldots, k. \tag{2}$$

This condition involves taking k dot products, one for each X_i. We can do them all at once by setting up a matrix A using the X_i as the columns of A, that is, let

$$A = \left[\begin{array}{c|c|c|c} X_1 & X_2 & \cdots & X_k \end{array}\right].$$

Note that each vector $X_i \in \mathbb{R}^n$ has n coordinates, so that A is an $n \times k$ matrix. The set of conditions listed in (2) can now be re-written:

$$A^T (Y - \hat{Y}) = 0,$$

which is equivalent to

$$A^T Y = A^T \hat{Y}. \tag{3}$$

Meanwhile, we need the projected vector $\hat{Y}$ to be a vector in W, since we are projecting onto W. This means that $\hat{Y}$ lies in the span of the vectors $X_1, \ldots, X_k$. In other

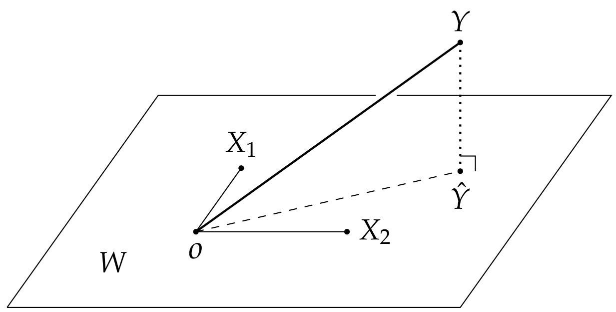

FIGURE 1. Projection of a vector onto a subspace.

words,

$$\hat{Y} = c_1X_1 + c_2X_2 + \cdots + c_kX_k = A \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = AC.$$

where C is a k-dimensional column vector. On combining this with the matrix equation (3) we have

$$A^TY = A^TAC.$$

If we knew what C was then we would also know $\hat{Y}$, since we were given the columns X_i of A, and $\hat{Y} = AC$. To solve for C just invert the $k \times k$ matrix A^TA to get

$$(A^TA)^{-1}A^TY = C. \tag{4}$$

How do we know that $(A^TA)^{-1}$ exists? Let's assume it does for now, and then address this question later on.

Now finally we can find our projected vector $\hat{Y}$. Since $\hat{Y} = AC$, multiply both sides of (4) to obtain

$$A(A^TA)^{-1}A^TY = AC = \hat{Y}.$$

The matrix

$$Q = A(A^TA)^{-1}A^T$$

is called the *projection matrix for the subspace* W. According to our derivation above, the projection matrix Q maps a vector $Y \in \mathbb{R}^n$ to its orthogonal projection (i.e. its shadow) $QY = \hat{Y}$ in the subspace W.

It is easy to check that Q has the following nice properties:

(1) $Q^T = Q$.
(2) $Q^2 = Q$.

One can show that any matrix satisfying these two properties is in fact a projection matrix for its own column space. You can prove this using the hints given in the exercises.

There remains one problem. At a crucial step in the derivation above we took the inverse of the $k \times k$ matrix $A^T A$. But how do we know this matrix is invertible? It is invertible, because the columns of A, the vectors $X_1, \ldots, X_k$, were assumed to be *linearly independent*. But this claim of invertibility needs a proof.

Lemma 1. *Suppose A is an $n \times k$ matrix, where $k \le n$, such that the columns of A are linearly independent. Then the $k \times k$ matrix $A^T A$ is invertible.*

Proof of Lemma 1: Suppose that $A^T A$ is *not* invertible. In this case, there exists a vector $X \neq 0$ such that $A^T A X = 0$. It then follows that

$$(AX) \cdot (AX) = (AX)^T AX = X^T A^T AX = X^T 0 = 0,$$

so that the length $\|AX\| = \sqrt{(AX) \cdot (AX)} = 0$. In other words, the length of AX is zero, so that $AX = 0$. Since $X \neq 0$, this implies that the columns of A are linearly dependent. Therefore, if the columns of A are linearly independent, then $A^T A$ must be invertible. $\square$

Example: Compute the projection matrix Q for the 2-dimensional subspace W of $\mathbb{R}^4$ spanned by the vectors $(1, 1, 0, 2)$ and $(-1, 0, 0, 1)$. What is the orthogonal projection of the vector $(0, 2, 5, -1)$ onto W?

Solution: Let

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 0 \\ 2 & 1 \end{bmatrix}.$$

Then

$$A^T A = \begin{bmatrix} 6 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad (A^T A)^{-1} = \begin{bmatrix} \frac{2}{11} & \frac{-1}{11} \\ \frac{-1}{11} & \frac{6}{11} \end{bmatrix},$$

so that the projection matrix Q is given by

$$Q = A(A^T A)^{-1} A^T = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{11} & \frac{-1}{11} \\ \frac{-1}{11} & \frac{6}{11} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 2 \\ -1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{10}{11} & \frac{3}{11} & 0 & \frac{-1}{11} \\ \frac{3}{11} & \frac{2}{11} & 0 & \frac{3}{11} \\ 0 & 0 & 0 & 0 \\ \frac{-1}{11} & \frac{3}{11} & 0 & \frac{10}{11} \end{bmatrix}$$

We can now compute the orthogonal projection of the vector $(0, 2, 5, -1)$ onto W. This is

$$\begin{bmatrix} \frac{10}{11} & \frac{3}{11} & 0 & \frac{-1}{11} \\ \frac{3}{11} & \frac{2}{11} & 0 & \frac{3}{11} \\ 0 & 0 & 0 & 0 \\ \frac{-1}{11} & \frac{3}{11} & 0 & \frac{10}{11} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 5 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{7}{11} \\ \frac{1}{11} \\ 0 \\ \frac{-4}{11} \end{bmatrix}$$

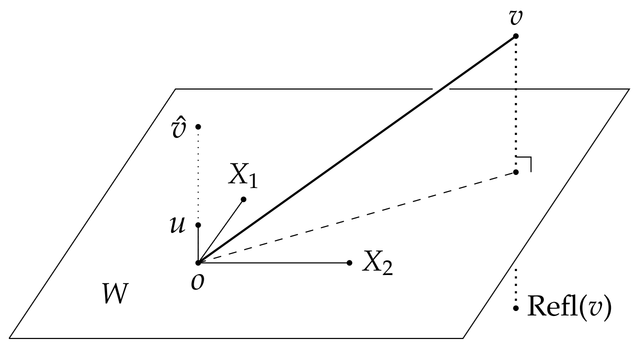

FIGURE 2. Reflection of the vector v across the plane W.

Reflections

We have seen earlier in the course that reflections of space across (i.e. through) a plane is linear transformation. Like rotations, a reflection preserves lengths and angles, although, unlike rotations, a reflection reverses orientation ("handedness").

Once we have projection matrices it is easy to compute the matrix of a reflection. Let W denote a plane passing through the origin, and suppose we want to reflect a vector v across this plane, as in Figure 2.

Let u denote a unit vector along $W^\perp$, that is, let u be a normal to the plane W. We will think of u and v as column vectors. The projection of v along the line through u is then given by:

$$\hat{v} = Proj_u(v) = u(u^T u)^{-1} u^T v.$$

But since we chose u to be a *unit* vector, $u^T u = u \cdot u = 1$, so that

$$\hat{v} = Proj_u(v) = uu^T v.$$

Let Q_u denote the matrix uu^T, so that $\hat{v} = Q_u v$.

What is the reflection of v across W? It is the vector $\text{Refl}_W(v)$ that lies on the other side of W from v, exactly the same distance from W as is v, and having the same projection into W as v. See Figure 2. The distance between v and its reflection is exactly twice the distance of v to W, and the difference between v and its reflection is perpendicular to W. That is, the difference between v and its reflection is exactly twice the projection of v along the unit normal u to W. This observation yields the equation:

$$v - \text{Refl}_W(v) = 2Q_u v,$$

so that

$$\text{Refl}_W(v) = v - 2Q_u v = Iv - 2Q_u v = (I - 2uu^T)v.$$

The matrix $H_W = I - 2uu^T$ is called the *reflection matrix for the plane* W, and is also sometimes called a *Householder matrix*.

Example: Compute the reflection of the vector $v = (-1, 3, -4)$ across the plane $2x - y + 7z = 0$.

Solution: The vector $w = (2, -1, 7)$ is normal to the plane, and $w^T w = 2^2 + (-1)^2 + 7^2 = 54$, so a unit normal will be

$$u = \frac{w}{|w|} = \frac{1}{\sqrt{54}}(2, -1, 7).$$

The reflection matrix is then given by

$$H = I - 2uu^T = I - \frac{2}{54}ww^T = I - \frac{1}{27}\begin{bmatrix} 2 \\ -1 \\ 7 \end{bmatrix}\begin{bmatrix} 2 & -1 & 7 \end{bmatrix} = \cdots$$

$$\cdots = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{27}\begin{bmatrix} 4 & -2 & 14 \\ -2 & 1 & -7 \\ 14 & -7 & 49 \end{bmatrix},$$

so that

$$H = \begin{bmatrix} 23/27 & 2/27 & -14/27 \\ 2/27 & 26/27 & 7/27 \\ -14/27 & 7/27 & -22/27 \end{bmatrix}$$

The reflection of v across W is then given by

$$Hv = \begin{bmatrix} 23/27 & 2/27 & -14/27 \\ 2/27 & 26/27 & 7/27 \\ -14/27 & 7/27 & -22/27 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 39/27 \\ 48/27 \\ 123/27 \end{bmatrix}$$

Reflections and Projections

Notice in Figure 2 that the *projection* of v into W is the *midpoint* of the vector v and its reflection $Hv = \mathrm{Refl}_W(v)$; that is,

$$Qv = \frac{1}{2}(v + Hv) \quad \text{or, equivalently} \quad Hv = 2Qv - v,$$

where $Q = Q_W$ denotes the projection onto W. (This is not the same as Q_u in the previous section, which was projection onto the *normal* of W.)

Let I denote the identity matrix, so that $v = Iv$ for all vectors $v \in \mathbb{R}^n$. The identities above can now be expressed as matrix identities:

$$Q = \frac{1}{2}(I + H) \quad \text{and} \quad H = 2Q - I,$$

So once you have computed the either the projection or reflection matrix for a subspace of $\mathbb{R}^n$, the other is quite easy to obtain.

Exercises

1. Suppose that M is an $n \times n$ matrix such that $M^T = M = M^2$. Let W denote the column space of M.

(a) Suppose that $Y \in W$. (This means that $Y = MX$ for some X.) Prove that $MY = Y$.
(b) Suppose that v is a vector in $\mathbb{R}^n$. Why is $Mv \in W$?
(c) If $Y \in W$, why is $v - Mv \perp Y$?
(d) Conclude that Mv is the projection of v into W.

2. Compute the projection of the vector $v = (1, 1, 0)$ onto the plane $x + y - z = 0$.

3. Compute the projection matrix Q for the subspace W of $\mathbb{R}^4$ spanned by the vectors $(1, 2, 0, 0)$ and $(1, 0, 1, 1)$.

4. Compute the orthogonal projection of the vector $z = (1, -2, 2, 2)$ onto the subspace W of Problem **3.** above. What does your answer tell you about the relationship between the vector z and the subspace W?

5. Recall that a square matrix P is said to be an *orthogonal matrix* if $P^T P = I$. Show that Householder matrices are always orthogonal matrices; that is, show that $H^T H = I$.

6. Compute the Householder matrix for reflection across the plane $x + y - z = 0$.

7. Compute the reflection of the vector $v = (1, 1, 0)$ across the plane $x + y - z = 0$. What happens when you add v to its reflection? How does this sum compare to your answer from Exercise 2? Draw a sketch to explain this phenomenon.

8. Compute the reflection of the vector $v = (1, 1)$ across the line ℓ in $\mathbb{R}^2$ spanned by the vector $(2, 3)$. Sketch the vector v, the line ℓ and the reflection of v across ℓ. (Do not confuse the spanning vector for ℓ with the normal vector to ℓ.)

9. Compute the Householder matrix H for reflection across the hyperplane $x_1 + 2x_2 - x_3 - 3x_4 = 0$ in $\mathbb{R}^4$. Then compute the projection matrix Q for this hyperplane.

10. Compute the Householder matrix for reflection across the plane $z = 0$ in $\mathbb{R}^3$. Sketch the reflection involved. Your answer should not be too surprising!

Selected Solutions to Exercises:

2. We describe two ways to solve this problem.

Solution 1: Pick a basis for the plane. Since the plane is 2-dimensional, any two independent vectors in the plane will do, say, $(1,-1,0)$ and $(0,1,1)$. Set

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A^T A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

The projection matrix Q for the plane is

$$Q = A(A^T A)^{-1} A^T = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 & 1/3 \\ -1/3 & 2/3 & 1/3 \\ 1/3 & 1/3 & 2/3 \end{bmatrix}$$

We can now project any vector onto the plane by multiplying by Q:

$$Projection(v) = Qv = \begin{bmatrix} 2/3 & -1/3 & 1/3 \\ -1/3 & 2/3 & 1/3 \\ 1/3 & 1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ 2/3 \end{bmatrix}$$

Solution 2: First, project v onto the normal vector $n = (1,1,-1)$ to the plane:

$$y = Proj_n(v) = \frac{v \cdot n}{n \cdot n} n = (2/3, 2/3, -2/3).$$

Since y is the component of v orthogonal to the plane, the vector $v - y$ is the orthogonal projection of v onto the plane. The solution (given in row vector notation) is

$$v - y = (1,1,0) - (2/3, 2/3, -2/3) = (1/3, 1/3, 2/3),$$

as in the previous solution.

Note: Which method is better? The second way is shorter for hyperplanes (subspaces of $\mathbb{R}^n$ having dimension $n-1$), but finding the projection matrix Q is needed if you are projecting from $\mathbb{R}^n$ to some intermediate dimension k, where you no longer have a single normal vector to work with. For example, a 2-subspace in $\mathbb{R}^4$ has a 2-dimensional orthogonal complement as well, so one must compute a projection matrix in order to project to either component of $\mathbb{R}^4$, as in the next problem.

3. Set

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{so that} \quad A^T A = \begin{bmatrix} 5 & 1 \\ 1 & 3 \end{bmatrix}$$

The projection matrix Q for the plane is

$$Q = A(A^TA)^{-1}A^T = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{14} & -\frac{1}{14} \\ -\frac{1}{14} & \frac{5}{14} \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{6}{14} & \frac{4}{14} & \frac{4}{14} & \frac{4}{14} \\ \frac{4}{14} & \frac{12}{14} & -\frac{2}{14} & -\frac{2}{14} \\ \frac{4}{14} & -\frac{2}{14} & \frac{5}{14} & \frac{5}{14} \\ \frac{4}{14} & -\frac{2}{14} & \frac{5}{14} & \frac{5}{14} \end{bmatrix}$$

5. Here is a hint: Use the fact that $H = I - 2uu^T$, where I is the identity matrix and u is a unit column vector. What is $H^T =$? What is $H^TH =$?

6. The vector $v = (1, 1, -1)$ is normal to the plane $x + y - z = 0$, so the vector $u = \frac{1}{\sqrt{3}}(1, 1, -1) = \frac{1}{\sqrt{3}}v$ is a unit normal. Expressing u and v as column vectors we find that

$$I - 2uu^T = I - (2/3)vv^T = I - \frac{2}{3}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$