Lectures notes on rotations (with exercises)
92.222 - Linear Algebra II - Spring 2005
by D. Klain

1. How to compute the orthogonal matrix that represents a rotation of $\mathbb{R}^3$

Recall that the 2×2 matrix

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

rotates the plane $\mathbb{R}^2$ counter-clockwise by the angle θ around the origin. Is there a similar way to represent rotations of 3-dimensional space using 3×3 matrices?

Consider the simple case of rotating 3-dimensional space by the same angle θ counter-clockwise around the z-axis. This is analogous to rotating the earth by the angle θ around the north pole, for example. This rotation fixes the z-axis, and acts on the xy-plane in the exactly the same way as the 2×2 matrix A_θ above. Therefore, the matrix of rotation around the z-axis by the counter-clockwise angle θ is given by

$$S_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where we assign the column vector $(1,0,0)^T$ to the x-axis, $(0,1,0)^T$ to the y-axis, and $(0,0,1)^T$ to the z-axis.

Note that, like A_θ, the matrix S_θ is an *orthogonal matrix*, that is,

$$S_\theta S_\theta^T = I \quad \text{or, equivalently,} \quad S_\theta^T = S_\theta^{-1}.$$

More generally, suppose we rotate 3-dimensional space counter-clockwise by the angle θ around a *different* axis through the origin, pointing along the direction of some unit vector $\mathbf{u} \in \mathbb{R}^3$. For this we need the analogue of the matrix S_θ, for which the z-axis is replaced by a different axis of rotation, the line passing through the point $\mathbf{u}$ and the origin o. Let us call this new rotation matrix $R_{\theta,\mathbf{u}}$, depending as it does on both the choice of axis $\mathbf{u}$ and the angle of rotation θ.

To compute $R_{\theta,\mathbf{u}}$, choose a unit vector $\mathbf{v}$ that is orthogonal to $\mathbf{u}$; that is, so that $\mathbf{u} \cdot \mathbf{v} = 0$. Let $\mathbf{w} = \mathbf{u} \times \mathbf{v}$, where $\times$ denotes the vector cross product in $\mathbb{R}^3$. We now have a new orthonormal basis for $\mathbb{R}^3$, $\{\mathbf{v}, \mathbf{w}, \mathbf{u}\}$ such that $\mathbf{v} \times \mathbf{w} = \mathbf{u}$. (It might help the reader to sketch this basis, where $\mathbf{u}$ is the vector pointing upwards in your picture, in analogy to the z-axis.)

Let P denote the matrix having $\mathbf{v}, \mathbf{w}, \mathbf{u}$ as its three columns (in that exact order):

$$P = \left[\begin{array}{c|c|c} \mathbf{v} & \mathbf{w} & \mathbf{u} \end{array}\right], \tag{1}$$

Note that P is an orthogonal matrix, $P^T P = I$, since the columns of P were (deliberately) chosen to form an orthonormal set. Note in particular that

$$P^T\mathbf{u} = \begin{bmatrix} \mathbf{v}^T \\ \hline \mathbf{w}^T \\ \hline \mathbf{u}^T \end{bmatrix} \mathbf{u} = \begin{bmatrix} \mathbf{v}^T\mathbf{u} \\ \mathbf{w}^T\mathbf{u} \\ \mathbf{u}^T\mathbf{u} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{2}$$

and that, similarly,

$$P^T\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad P^T\mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Theorem 1 *The matrix $R_{\theta,\mathbf{u}}$ that rotates $\mathbb{R}^3$ around the vector $\mathbf{u}$ by the counterclockwise angle θ is given by the formula*

$$R_{\theta,\mathbf{u}} = PS_\theta P^T \tag{3}$$

Proof of Theorem 1: To begin, consider what the transformation $PS_\theta P^T$ does to the vectors $\mathbf{v}, \mathbf{w}, \mathbf{u}$. The matrix $PS_\theta P^T$ *fixes* $\mathbf{u}$; indeed, by (1) and (2),

$$PS_\theta P^T\mathbf{u} = PS_\theta \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = P \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{u}.$$

Similarly,

$$PS_\theta P^T\mathbf{v} = PS_\theta \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = P \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} = \cos\theta\,\mathbf{v} + \sin\theta\,\mathbf{w},$$

while

$$PS_\theta P^T\mathbf{w} = PS_\theta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = P \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} = -\sin\theta\,\mathbf{v} + \cos\theta\,\mathbf{w}.$$

More generally, if $X = a\mathbf{v} + b\mathbf{w} + c\mathbf{u}$ is any vector in $\mathbb{R}^3$ (expressed in terms of the orthonormal basis $\{\mathbf{v}, \mathbf{w}, \mathbf{u}\}$) then

$$\begin{aligned} PS_\theta P^T X &= aPS_\theta P^T\mathbf{v} + bPS_\theta P^T\mathbf{w} + cPS_\theta P^T\mathbf{u} \\ &= a(\cos\theta\,\mathbf{v} + \sin\theta\,\mathbf{w}) + b(-\sin\theta\,\mathbf{v} + \cos\theta\,\mathbf{w}) + c\mathbf{u} \quad = \quad R_{\theta,\mathbf{u}}X, \end{aligned}$$

rotating X counterclockwise by θ in the $\mathbf{vw}$-plane orthogonal to the axis of rotation $\mathbf{u}$. ∎

The identity (3), together with the orthogonality of P and S_θ, implies that $R_{\theta,\mathbf{u}}$ is also an orthogonal matrix. More precisely, we have the following corollary.

Corollary 2 *A rotation matrix R is an orthogonal matrix with determinant 1.*

Proof: If R is a rotation matrix then $R = PS_\theta P^T$, where $P^T = P^{-1}$ and $S_\theta^T = S_\theta^{-1}$, as in Theorem 1. Therefore,

$$R^T R = (PS_\theta P^T)^T PS_\theta P^T = PS_\theta^T P^T PS_\theta P^T = PS_\theta^T S_\theta P^T = PP^T = I,$$

so that R is an orthogonal matrix. Moreover,

$$\det(R) = \det(PS_\theta P^T) = \det(PS_\theta P^{-1}) = \det(P)\det(S_\theta)\frac{1}{\det(P)} = \det(S_\theta) = 1.$$

∎

Remark: The converse of the Corollary is also true: A matrix R is a rotation matrix *if and only if* R is an orthogonal matrix and $\det(R) = 1$. But we will not prove this now.

Example: Find the matrix $R_{\frac{\pi}{3},\mathbf{u}}$ that rotates $\mathbb{R}^3$ by the counterclockwise angle $\pi/3$ around the axis through the vector $\mathbf{u} = (2, 1, 1)$.

Solution: To begin, find a vector $\mathbf{v}$ that is perpendicular to $\mathbf{u} = (2, 1, 1)$. An easy choice is $\mathbf{v} = (0, 1, -1)$. We then set $\mathbf{w} = \mathbf{u} \times \mathbf{v}$, so that

$$\mathbf{w} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} = (-2, 2, 2).$$

We now have an *orthogonal* set $\{\mathbf{v}, \mathbf{w}, \mathbf{u}\}$, where $\mathbf{v} \times \mathbf{w} = \mathbf{u}$ and $\mathbf{u}$ is parallel to our desired axis of rotation. Unfortunately, however, the vectors $\mathbf{v}, \mathbf{w}, \mathbf{u}$ are *not unit vectors.* This is easily fixed: dividing each vector by its length, re-assign the variables $\mathbf{v}, \mathbf{w}, \mathbf{u}$ to form the *orthonormal set:*

$$\mathbf{v} = (0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) \quad \mathbf{w} = (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) \quad \mathbf{u} = (\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}),$$

so that

$$P = \left[\begin{array}{c|c|c} \mathbf{v} & \mathbf{w} & \mathbf{u} \end{array} \right] = \begin{bmatrix} 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

It now follows from Theorem 1 that

$$R_{\frac{\pi}{3},\mathbf{u}} = PS_{\frac{\pi}{3}}P^T = \begin{bmatrix} 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}.$$

After multiplying these matrices, we obtain

$$R_{\frac{\pi}{3},\mathbf{u}} = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} - \frac{1}{2\sqrt{2}} & \frac{1}{6} + \frac{1}{2\sqrt{2}} \\ \frac{1}{6} + \frac{1}{2\sqrt{2}} & \frac{7}{12} & \frac{1}{12} - \frac{1}{\sqrt{2}} \\ \frac{1}{6} - \frac{1}{2\sqrt{2}} & \frac{1}{12} + \frac{1}{\sqrt{2}} & \frac{7}{12} \end{bmatrix}. \tag{4}$$

2. How to compute the rotation of $\mathbb{R}^3$ represented by a given orthogonal matrix

Now suppose you are given an orthogonal matrix R such that $\det R = 1$; in other words, a rotation matrix. What is the axis of rotation for R? What is the angle of rotation? How do we compute $\mathbf{u}$ and θ so that $R = R_{\theta,\mathbf{u}}$?

Here is one quick test to find θ. Recall that the *trace* of a square $n \times n$ matrix A is the sum of its diagonal entries: $\mathrm{trace}(A) = A_{11} + A_{22} + \cdots + A_{nn}$.

Theorem 3 (The Cosine Test) *If R is a rotation matrix having angle of rotation θ, then*

$$\cos\theta = \frac{\mathrm{trace}(R) - 1}{2}. \tag{5}$$

Proof: We will need the fact that if A is any square $n \times n$ matrix, and P is an $n \times n$ invertible matrix, then $\mathrm{trace}(PAP^{-1}) = \mathrm{trace}(A)$. This is a consequence of the fact that, for any two $n \times n$ matrices A and B, we have $\mathrm{trace}(AB) = \mathrm{trace}(BA)$. (You can check this directly by using the matrix multiplication formula.)

If R is a rotation matrix having angle of rotation θ, then $R = R_{\theta,\mathbf{u}}$ for some unit vector $\mathbf{u}$, so that $R = PS_\theta P^T = PS_\theta P^{-1}$, as in (3). Hence,

$$\mathrm{trace}(R) = \mathrm{trace}(PS_\theta P^{-1}) = \mathrm{trace}(S_\theta) = 1 + 2\cos\theta,$$

from which the formula (5) above immediately follows. ∎

The Cosine Test, while very easy to use, doesn't tell the whole story, since the axis of rotation $\mathbf{u}$ remains unknown. Moreover, there remains an ambiguity regarding the value of θ, since we only know $\cos\theta$. Since $\cos\theta = \cos(-\theta)$, the sign of the angle remains obscure.

Fortunately it takes only a tiny bit of work to compute $\mathbf{u}$. The key is to remember that if $\mathbf{u}$ lies in the axis of rotation, then the rotation R fixes the vector $\mathbf{u}$. In other words, $R\mathbf{u} = \mathbf{u}$. Since the inverse matrix R^{-1} will represent rotation around the *same* axis $\mathbf{u}$ by the negative of the angle θ, we also have $R^{-1}\mathbf{u} = \mathbf{u}$. Recall that R is an orthogonal matrix, so that $R^T = R^{-1}$. It now follows that $R^T\mathbf{u} = R^{-1}\mathbf{u} = \mathbf{u}$, so that

$$(R - R^T)\mathbf{u} = R\mathbf{u} - R^T\mathbf{u} = \mathbf{u} - \mathbf{u} = 0.$$

This suggests that we can discover the vector $\mathbf{u}$ by considering the null space of the matrix $R - R^T$.

Denote

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

We then have

$$R - R^T = \begin{bmatrix} 0 & r_{12} - r_{21} & r_{13} - r_{31} \\ r_{21} - r_{12} & 0 & r_{23} - r_{32} \\ r_{31} - r_{13} & r_{32} - r_{23} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{bmatrix},$$

where we denote $\alpha = r_{12} - r_{21}$, $\beta = r_{13} - r_{31}$, and $\gamma = r_{23} - r_{32}$. This suggests that $\mathbf{u}$ is *parallel* to the vector

$$\mathbf{q} = \begin{bmatrix} -\gamma \\ \beta \\ -\alpha \end{bmatrix} = \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}, \tag{6}$$

assuming this vector is not the zero vector (which might happen sometimes).

We summarize this result, and make it more precise, with the following theorem.

Theorem 4 (The Symmetric Difference Test) *Suppose that R is a rotation matrix, and suppose that $R^T \neq R$, so that the vector $\mathbf{q} \neq 0$. Then the axis of rotation of R is parallel to $\mathbf{q}$. More specifically, the matrix R rotates $\mathbb{R}^3$ by a positive counterclockwise angle θ around the unit vector $\mathbf{u}$, where*

$$\mathbf{q} = 2(\sin\theta)\,\mathbf{u}.$$

Note, in particular, that $2\sin\theta = |\mathbf{q}|$ and $\mathbf{u} = \frac{\mathbf{q}}{|\mathbf{q}|}$. Using **both** Theorem 3 and Theorem 4 we obtain the axis of rotation, with direction and orientation provided by $\mathbf{u}$, and the exact value of the angle θ, from the values of $\cos\theta$ and $\sin\theta$.

Proof: Suppose that $R = R_{\theta,\mathbf{u}} = PS_\theta P^T$ as in (3). Then

$$\begin{bmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{bmatrix} = R - R^T = PS_\theta P^T - PS_\theta^T P^T = P(S_\theta - S_\theta^T)P^T = P \begin{bmatrix} 0 & -2\sin\theta & 0 \\ 2\sin\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} P^T$$

$$= \left[\begin{array}{c|c|c} \mathbf{v} & \mathbf{w} & \mathbf{u} \end{array}\right] \begin{bmatrix} 0 & -2\sin\theta & 0 \\ 2\sin\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}^T \\ \hline \mathbf{w}^T \\ \hline \mathbf{u}^T \end{bmatrix} = 2(\sin\theta)(\mathbf{w}\mathbf{v}^T - \mathbf{v}\mathbf{w}^T),$$

so that

$$\alpha = 2\sin\theta(v_2 w_1 - v_1 w_2), \quad \beta = 2\sin\theta(v_3 w_1 - v_1 w_3), \quad \gamma = 2\sin\theta(v_3 w_2 - v_2 w_3).$$

In other words,

$$\mathbf{q} = \begin{bmatrix} -\gamma \\ \beta \\ -\alpha \end{bmatrix} = 2(\sin\theta)\,\mathbf{v}\times\mathbf{w} = 2(\sin\theta)\,\mathbf{u}.$$

∎

Example: Let's use Theorem 4 to check the work we did in the last example, where

$$R = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} - \frac{1}{2\sqrt{2}} & \frac{1}{6} + \frac{1}{2\sqrt{2}} \\ \frac{1}{6} + \frac{1}{2\sqrt{2}} & \frac{7}{12} & \frac{1}{12} - \frac{1}{\sqrt{2}} \\ \frac{1}{6} - \frac{1}{2\sqrt{2}} & \frac{1}{12} + \frac{1}{\sqrt{2}} & \frac{7}{12} \end{bmatrix}.$$

In this case, we use (6) to compute

$$\mathbf{q} = \begin{bmatrix} \frac{2}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix},$$

so that $2\sin\theta = |\mathbf{q}| = \sqrt{3}$. This implies that $\theta = \arcsin(\frac{\sqrt{3}}{2}) = \frac{\pi}{3}$, and that $\mathbf{u}$ is parallel to $(2, 1, 1)$, as we began with in the previous example.

We can double-check the angle calculation with the Cosine Test. In this case, we have

$$\cos\theta = \frac{\mathrm{trace}(R) - 1}{2} = \frac{1}{2}\left(\frac{5}{6} + \frac{7}{12} + \frac{7}{12} - 1\right) = \frac{1}{2},$$

so that $\theta = \arccos(\frac{1}{2}) = \frac{\pi}{3}$ once again.

Question: Theorem 4 assumes that $R \neq R^T$. What if $R = R^T$? In this case we get $R - R^T = 0$, the zero matrix, so that $\mathbf{q} = 0$, the zero vector. From this we can deduce that $\sin\theta = 0$, so that either $\theta = 0$ or $\theta = \pi$. If $\theta = 0$, then R is the identity rotation, and this would be obvious immediately, since R would be the identity matrix! So if $R \neq I$ we know that $\theta = \pi$. But what is the axis of rotation? Since $\theta = \pi$ in this instance, we have

$$R = PS_\pi P^T = P \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} P^T = -\mathbf{v}\mathbf{v}^T - \mathbf{w}\mathbf{w}^T + \mathbf{u}\mathbf{u}^T.$$

Since $\mathbf{v}, \mathbf{w}, \mathbf{u}$ form an orthonormal basis, $\mathbf{v}\mathbf{v}^T + \mathbf{w}\mathbf{w}^T + \mathbf{u}\mathbf{u}^T = I$, the identity matrix (Why?), so that

$$R = -\mathbf{v}\mathbf{v}^T - \mathbf{w}\mathbf{w}^T - \mathbf{u}\mathbf{u}^T + 2\mathbf{u}\mathbf{u}^T = -I + 2\mathbf{u}\mathbf{u}^T,$$

and $2\mathbf{u}\mathbf{u}^T = I + R$. But the columns of the matrix $\mathbf{u}\mathbf{u}^T$ are each parallel to $\mathbf{u}$ (Why?), so the vector $\mathbf{u}$ can be obtained by taking any non-zero column of $I + R$ and normalizing to a unit vector.

3. Summary

To compute $R_{\theta,\mathbf{u}}$ from a **unit** vector $\mathbf{u}$ and an angle θ:

(1) Choose any unit vector $\mathbf{v}$ such that $\mathbf{v} \perp \mathbf{u}$.

(2) Set $\mathbf{w} = \mathbf{u} \times \mathbf{v}$ and set $P = \left[\begin{array}{c|c|c} \mathbf{v} & \mathbf{w} & \mathbf{u} \end{array}\right]$.

(3) The matrix $R_{\theta,\mathbf{u}}$ is given by

$$R_{\theta,\mathbf{u}} = PS_\theta P^T = \left[\begin{array}{c|c|c} \mathbf{v} & \mathbf{w} & \mathbf{u} \end{array}\right] \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{c} \mathbf{v}^T \\ \hline \mathbf{w}^T \\ \hline \mathbf{u}^T \end{array}\right].$$

To compute $\mathbf{u}$ and θ from a rotation matrix R:

(1) If $R \neq R^T$, then set $\mathbf{q} = \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$.

In this case $R = R_{\theta,\mathbf{u}}$ where $\mathbf{u} = \mathbf{q}/|\mathbf{q}|$ and $\sin\theta = |\mathbf{q}|/2$, and $\cos\theta = \frac{trace(R)-1}{2}$.

(2) If $R = R^T$ and $R \neq I$ then $R = R_{\mathbf{u},\pi}$ where $\mathbf{u}$ is a unit vector parallel to any non-zero column of $I + R$.

(3) If $R = I$ then R is the identity rotation (angle zero, everything stays fixed).

Exercises:

1. Compute the matrix $R_{\frac{\pi}{4},(1,1,1)}$.

2. Compute the matrix $R_{\frac{\pi}{6},(0,1,0)}$.

3. Compute the matrix $R_{\pi,(2,0,1)}$.

4. Compute the matrix $R_{2\pi,(2,0,1)}$.

5. Compute the angle θ and axis of rotation $\mathbf{u}$ for the rotation matrix

$$R = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}.$$

6. Compute the angle θ and axis of rotation $\mathbf{u}$ for the rotation matrix

$$R = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{6}} & \frac{2}{\sqrt{30}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{30}} \\ 0 & \frac{1}{\sqrt{6}} & -\frac{5}{\sqrt{30}} \end{bmatrix}.$$

7. Compute the angle θ and axis of rotation $\mathbf{u}$ for the rotation matrix

$$R = \begin{bmatrix} -\frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}.$$

8. Suppose that $R = R_{\theta,\mathbf{u}}$. Prove that $R^{-1} = R_{-\theta,\mathbf{u}}$.

9. Suppose that $R = R_{\theta,\mathbf{u}}$. Prove that $R^T = R_{-\theta,\mathbf{u}}$.

10. Suppose that $R = R_{\theta,\mathbf{u}}$. Prove that $R^2 = R_{2\theta,\mathbf{u}}$.

11. Prove that $R_{-\theta,\mathbf{u}} = R_{\theta,-\mathbf{u}}$.

12. Suppose that $R = R_{\pi,\mathbf{u}}$. Prove that $(I + R)\mathbf{v} = 0$ and that $(I + R)\mathbf{w} = 0$.

Selected Solutions:

1. $R_{\frac{\pi}{4},(1,1,1)} = \begin{bmatrix} \frac{1}{3}+\frac{2}{3\sqrt{2}} & \frac{1}{3}-\frac{1}{\sqrt{6}}-\frac{1}{3\sqrt{2}} & \frac{1}{3}+\frac{1}{\sqrt{6}}-\frac{1}{3\sqrt{2}} \\ \frac{1}{3}+\frac{1}{\sqrt{6}}-\frac{1}{3\sqrt{2}} & \frac{1}{3}+\frac{2}{3\sqrt{2}} & \frac{1}{3}-\frac{1}{\sqrt{6}}-\frac{1}{3\sqrt{2}} \\ \frac{1}{3}-\frac{1}{\sqrt{6}}-\frac{1}{3\sqrt{2}} & \frac{1}{3}+\frac{1}{\sqrt{6}}-\frac{1}{3\sqrt{2}} & \frac{1}{3}+\frac{2}{3\sqrt{2}} \end{bmatrix}$.

2. $R_{\frac{\pi}{6},(0,1,0)} = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix}$.

3. $R_{\pi,(2,0,1)} = \begin{bmatrix} \frac{3}{5} & 0 & \frac{4}{5} \\ 0 & -1 & 0 \\ \frac{4}{5} & 0 & -\frac{3}{5} \end{bmatrix}$.

4. $R_{2\pi,(2,0,1)} = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

5. $\mathbf{q} = \begin{bmatrix} \frac{-\sqrt{2}-2}{\sqrt{6}} \\ \frac{-\sqrt{3}-1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix}$, and $R = R_{\mathbf{u},\theta}$, where $\mathbf{u} = \frac{\mathbf{q}}{|\mathbf{q}|} \approx \begin{bmatrix} -0.743 \\ -0.594 \\ -0.308 \end{bmatrix}$

and $\theta = \arcsin(\frac{|\mathbf{q}|}{2}) = \arccos(\frac{trace(R)-1}{2}) \approx 1.217$ radians.

6. $\mathbf{q} = \begin{bmatrix} \frac{1}{\sqrt{6}}+\frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{30}} \\ \frac{2}{\sqrt{5}}-\frac{2}{\sqrt{6}} \end{bmatrix}$, and $R = R_{\mathbf{u},\theta}$, where $\mathbf{u} = \frac{\mathbf{q}}{|\mathbf{q}|} \approx \begin{bmatrix} 0.845 \\ 0.522 \\ 0.111 \end{bmatrix}$

and $\theta = \arcsin(\frac{|\mathbf{q}|}{2}) = \arccos(\frac{trace(R)-1}{2}) \approx 2.785$ radians.

7. Since $R^T = R$ and $R \neq I$, it follows that $\theta = \pi$. We then compute

$$I + R = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{4}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix},$$

so that $\mathbf{u} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix}$ and $R = R_{\pi,(\frac{1}{\sqrt{6}},-\frac{2}{\sqrt{6}},-\frac{1}{\sqrt{6}})}$.