22.520 Numerical Methods for PDEs : *Video 21: The Weak Form*

David J. Willis

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References and Acknowledgements

The following materials were used in the preparation of this lecture:

1. 16.920, Lecture Notes
2. Strang & Fix: An analysis of the Finite Element Method
3. Bathe, Hughes, etc. have introductory books that are useful.
4. : Additional link to MIT 19.901, Undergraduate Numerical Methods Course – may be useful.

http://ocw.mit.edu/courses/aeronautics-and-astronautics/16-901-computational-methods-in-aerospace-engineering-spring-2005/ The author of these slides wishes to thank these sources for making the current lecture.

Weak Form

- In finite elements we consider the *weak form* of the governing equation.
- Consider a general PDE, represented by the functional $\mathcal{L}(u)$:

$$\mathcal{L}(u) = f \tag{1}$$

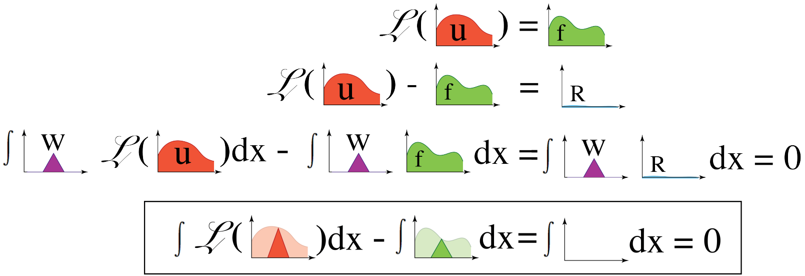
- With some boundary conditions $u_{\Gamma_D} = g$ and $\frac{\partial u}{\partial n}|_{\Gamma_N} = h$
- What if we multiply this by a function w and integrate across the domain of interest:

$$\int_{x_L}^{x_R} w\mathcal{L}(u)dx = \int_{x_L}^{x_R} wfdx \tag{2}$$

- Solving this expression is effectively the same as solving the original equation.
- **BUT** what does it mean to do this?

Weak Form

- Graphically, it shows that we are satisfying a weighted expression of the equation in the domain/sub-domain:



- Since we are satisfying a weighted form of the equation, we can accentuate the expression as desired.

Poisson's Equation?

- Let's do an example with the Poisson equation/Laplace operator in one dimension:

$$\frac{\partial^2 u}{\partial x^2} = f \tag{3}$$

- Let's re-arrange the equation a little:

$$\frac{\partial^2 u}{\partial x^2} - f = R = 0 \tag{4}$$

- This shows that the equation has a *zero residual,* R when solved exactly.

Weak Form

- Let's now multiply the expression by w, a weighting function and integrate across the domain/region of interest:

$$\int_{x_L}^{x_R} \left(w \frac{\partial^2 u}{\partial x^2} \right) dx - \int_{x_L}^{x_R} (wf)\, dx = \int_{x_L}^{x_R} wR dx = 0 \qquad (5)$$

- The result is an integral expression over the domain.
- Notice, that the *residual* has been weighted and integrated – and remains zero.

The weak form

- Let's integrate the weighted residual integral expression by parts:

$$\int_{x_L}^{x_R} \left(w \frac{\partial^2 u}{\partial x^2} \right) dx - \int_{x_L}^{x_R} (w f)\, dx = 0 \tag{6}$$

- Recall, integration by parts says:

$$\int_{x_L}^{x_R} f(x) g'(x) dx = f(x) g(x)|_{x_L}^{x_R} - \int_{x_L}^{x_R} f'(x) g(x) dx \tag{7}$$

- Let's set $f(x) = w$ and $g'(x) = \frac{\partial^2 u}{\partial x^2}$, then:

$$\int_{x_L}^{x_R} \left(w \frac{\partial^2 u}{\partial x^2} \right) dx = w \frac{\partial u}{\partial x}|_{x_L}^{x_R} - \int_{x_L}^{x_R} \frac{\partial w}{\partial x} \frac{\partial u}{\partial x} dx \tag{8}$$

The weak form

- Re-inserting this into the equation 6:

$$\int_{x_L}^{x_R} \left(w \frac{\partial^2 u}{\partial x^2} \right) dx \quad - \quad \int_{x_L}^{x_R} (wf)\, dx =$$
$$w \frac{\partial u}{\partial x}\Big|_{x_L}^{x_R} - \int_{x_L}^{x_R} \frac{\partial w}{\partial x}\frac{\partial u}{\partial x} dx \quad - \quad \int_{x_L}^{x_R} (wf)\, dx = 0$$

- Let's focus only on the reformulated expression:

$$w \frac{\partial u}{\partial x}\Big|_{x_L}^{x_R} - \int_{x_L}^{x_R} \frac{\partial w}{\partial x}\frac{\partial u}{\partial x} dx - \int_{x_L}^{x_R} (wf)\, dx = 0 \qquad (9)$$

- Which we will call the **weak form**.

The weak form

- The final weak form of the equation is:

$$\underbrace{w\frac{\partial u}{\partial x}\Big|_{x_L}^{x_R}}_{at\ \ boundaries} - \underbrace{\int_{x_L}^{x_R}\frac{\partial w}{\partial x}\frac{\partial u}{\partial x}dx}_{a(u,w)} - \underbrace{\int_{x_L}^{x_R} wfdx}_{f(w)} = 0 \tag{10}$$

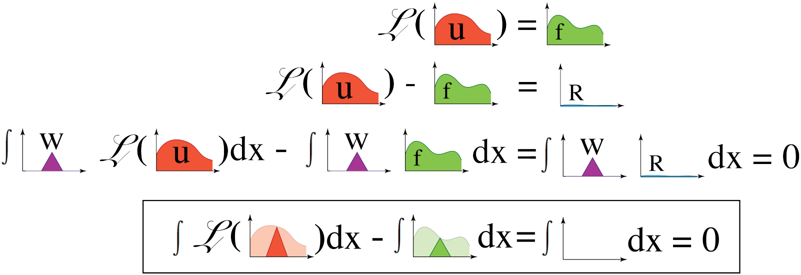
- $a(w, u)$: Is the bilinear form
- Often this is written as:

$$a(u, w) - f(w) = 0 \tag{11}$$

- This final weak form is integrals of first derivatives, whereas the strong form was second derivatives.

What have we learned

- Let's look at the graphic again:



- Integration by parts results in a reduction of order of PDE.