MECH 5810 Module 3: Conservation of Linear Momentum

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MECH 5810 Advanced Fluid Dynamics
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Outline

1. Announcements
2. Dynamics: Origin of Forces In Fluids
   - Volume Forces
   - Surface Forces
   - Line Forces
3. Integral Conservation of Linear Momentum
   - Material volumes
   - General conservation law for material volumes
   - General transport equation for control volumes

Announcements

- Homework # 1 is posted online.
- Any questions?

Outline

Fluid Motion 101

- Newton's Laws will govern the motion of a fluid.
- Today's key concept:

$$\sum \vec{F} = \frac{\partial(momentum)}{\partial t} = \frac{\partial(m\vec{u})}{\partial t} = \underbrace{\vec{u}\frac{\partial m}{\partial t}}_{Changing \quad Mass} + \underbrace{m\vec{a}}_{acceleration} \tag{1}$$

- First: The forces acting on a control volume.
- Second: The resulting change in momentum of the fluid.

Formalizing External Forces: Body Forces

- *Body* or *Volume* Forces: result from the fluid being placed in a *force field*:
  - Gravity (Conservative)
  - Electrostatic (Conservative)
  - Magnetic (Conservative)
  - Electromagnetic
- Conservative forces:
  - Work done by the force is independent of the path
  - $\sum(PE + KE) = Const.$
- The body force acts on the volume of fluid (hence does not "touch" the fluid, but acts "on" the fluid)
- eg:

$$\vec{F}(t)_{vol} = \iiint_{MV(t)} \rho \vec{g} dV$$

Formalizing External Forces: Body Forces

- Representing body forces:

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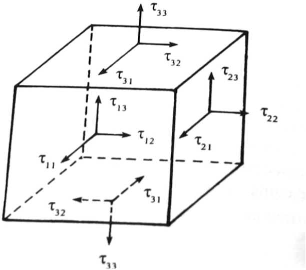
Formalizing External Forces: Surface Forces

- Surface forces: Act by direct contact on an area
- Expressed as "per area" quantities
  - Pressure (Normal force per unit area) $\rightarrow F_N = \iint -p \cdot d\vec{A}$
  - Shear stress (Tangential force per unit area) $\rightarrow F_t = \iint \tau dA$
- 9 components make up the stress tensor at a given point (more later in the course):



Formalizing External Forces: Surface Forces

- Representing surface forces due to normal stress/pressure:

Formalizing External Forces: Surface Forces

- Representing surface forces due to shear stresses:

Outline

Formalizing External Forces: Line Forces

- Line forces: Act along a contact line.
- Expressed as "per length" quantities
  - Surface tension requires 2 fluids in contact.
  - Surface tension $\rightarrow F_\sigma = \int \sigma dl$
  - Surface tension acts perpendicular to a cut line.

Formalizing External Forces: Line Forces

- Representing line forces due to surface tension:

Formalizing External Forces: Putting it all together

- Determine the NET force on a volume of fluid.
- This is the force that will *produce* a momentum change.

Outline

Integral Conservation of Linear Momentum:Material Volume

- At **any** material point, $\frac{\partial(m\vec{u})}{\partial t} = m\vec{a} = \sum \vec{F}$ applies
- For an infinitesimal material volume:

$$m = \rho\delta V = \rho\delta x\delta y\delta z$$

- and:

$$\vec{a} = \frac{\partial\vec{u}}{\partial t}$$

- So, for a single material particle/infinitesimal volume:

$$m\vec{a} = (\rho\delta V)\left(\frac{\partial\vec{u}}{\partial t}\right) = \frac{\partial(\rho\vec{u}\delta V)}{\partial t} = \vec{F}$$

Integral Conservation of Linear Momentum: Material Volume

- To apply conservation of momentum over the entire *material volume*, we sum (integrate) the infinitesimal contributions of all material particles over the the volume:

$$\frac{d}{dt}\iiint_{MV(t)} \underbrace{\rho\vec{u}}_{momentum\ \ per\ \ vol} dV = \sum \vec{F}_{MV}(t)$$

- Recall, that the conservation of mass statement (for MV) is similar:

$$\frac{d}{dt}\iiint_{MV(t)} \rho dV = 0$$

Outline

Integral Conservation of Linear Momentum: Material Volume

- We can generalize this into a transport law for material volumes, which states:

$$\frac{dB}{dt} = \frac{d}{dt} \iiint_{MV(t)} \rho b dV$$

- Where b is the *per unit mass* representation of the quantity B.

Property	B	b
Mass	m_s	1
Linear Momentum	$m_s \vec{u}$	$\vec{u}$
Angular Momentum	$\vec{r} \times m_s \vec{u}$	$\vec{r} \times \vec{u}$
Kinetic Energy	$\frac{1}{2} m_s \|\|\vec{u}\|\|^2$	$\frac{\|\|\vec{u}\|\|^2}{2}$

Outline

Control Volume Integral Conservation Transport Equation

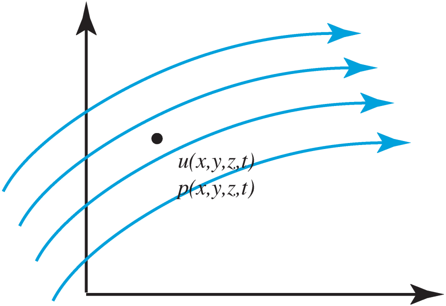
- Laws of physics are usually/commonly written for a fixed mass of material in a Lagrangian description (or Material Volume)
- We want to be able to calculate:

$$\frac{dB}{dt} = \iiint_{MV} \rho \frac{Db}{Dt} dV$$

  using an *Eulerian description* of the fluid.
- To do this, we must take a brief detour to understand how to calculate change in a property w.r.t. time.

ASIDE: Eulerian Description

- Eulerian description: examines fluid properties at an individual stationary point
- Independent variables: $\vec{r}'$ position in space, t' time.
  - A fluid property, F is expressed as $F(\vec{r}', t')$



ASIDE: Lagrangian Derivatives in an Eulerian Description(KC section 3.3)

- Lagrangian: time rate of change of a property $F(\vec{a}, t)$, is computed as we follow a material point.
- How do we compute this same **important** rate of change in an Eulerian or field description?
  - $\frac{\partial}{\partial t}[F(\vec{r}', t')] \neq \frac{\partial}{\partial t}[F(\vec{a}, t)]$
  - In the Lagrangian we are following the material particles, not staying fixed in space.
- So,

$$\left(\frac{\partial[F(\vec{r}', t')]}{\partial t}\right)_a = ?$$

- Ie. What is the rate of change of a property if we follow a particle, if we were to measure only at a fixed field point?

ASIDE: Lagrangian Derivatives in an Eulerian Description(KC section 3.3)

ASIDE: Lagrangian Derivatives in an Eulerian Description (KC section 3.3)

- Consider the property F when $\vec{r}(\vec{a}, t) = \vec{r}'$ and $t = t'$:

$$F(\vec{a}, t) = F[\vec{r}(\vec{a}, t)] = F(\vec{r}', t')$$

- Let's differentiate, careful to use the chain rule:

$$\left[\frac{\partial F(\vec{a}, t)}{\partial t}\right]_{\vec{a}} = \left(\frac{\partial F}{\partial t'}\right)_{\vec{r}'} \left(\frac{\partial t'}{\partial t}\right) + \left(\frac{\partial F}{\partial \vec{r}'}\right)_{t'} \cdot \left(\frac{\partial \vec{r}'}{\partial r}\right) \cdot \left(\frac{\partial \vec{r}}{\partial t}\right)_{\vec{a}}$$

- Review: Chain Rule

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \tag{2}$$

ASIDE: Lagrangian Derivatives in an Eulerian Description (KC section 3.3)

- Let's look term by term and simplify the equation:

$$\left[\frac{\partial F(\vec{a},t)}{\partial t}\right]_{\vec{a}} = \left(\frac{\partial F}{\partial t'}\right)_{\vec{r}'} \underbrace{\cancel{\left(\frac{\partial t'}{\partial t}\right)}}_{t=t'} + \underbrace{\left(\frac{\partial F}{\partial \vec{r}'}\right)_{t'}}_{spatial\ gradient\ of\ F} \cdot \underbrace{\cancel{\left(\frac{\partial \vec{r}'}{\partial r}\right)}}_{no\ frame\ rotation \rightarrow I} \cdot \underbrace{\left(\frac{\partial \vec{r}}{\partial t}\right)_{\vec{a}}}_{=\vec{u}}$$

- And, therefore:

$$\left(\frac{\partial F}{\partial t}\right)_a = \frac{\partial F}{\partial t'} + (\nabla' F) \cdot \vec{u} = \frac{DF}{Dt}$$

ASIDE: Lagrangian Derivatives in an Eulerian Description (KC section 3.3)

- The result is the *Material Derivative*:

$$\left(\frac{\partial F}{\partial t}\right)_a = \frac{\partial F}{\partial t} + \vec{u} \cdot (\nabla F) = \frac{DF}{Dt}$$

- Which is a combination of:
  1. The *local rate of change* of F w.r.t. time (ie. the change in F w.r.t. time as we sit at $\vec{r'}$):

$$\frac{\partial F}{\partial t}$$

  2. The *convective, or advective rate of change* of F w.r.t. time (ie. the change in F w.r.t. time due to the particle moving from one location to another, as measured in a field reference frame):

$$\vec{u} \cdot (\nabla F)$$

ASIDE: Lagrangian Derivatives in an Eulerian Description(KC section 3.3)

- Consider a fluid with a known temperature gradient flowing in a tank.

ASIDE: Lagrangian Derivatives in an Eulerian Description(KC section 3.3)

Control Volume Integral Conservation Transport Equation

- Returning to our integral conservation law expression.
- Recall: We want to be able to calculate:

$$\frac{dB}{dt} = \iiint_{MV} \rho \frac{Db}{Dt} dV$$

  using an *Eulerian description* of the fluid.
- Must invoke the *material derivative* for the acceleration term:

$$\frac{dB}{dt} = \iiint_{MV} \rho \left(\frac{\partial b}{\partial t} + (\vec{u} \cdot \nabla) b \right) dV$$

Control Volume Integral Conservation Transport Equation

- Mathematical manipulation of the material derivative:

$$
\begin{aligned}
\rho \frac{Db}{Dt} &= \rho \frac{\partial b}{\partial t} + \rho \left(\vec{u} \cdot \nabla \right) b \\
&= \rho \frac{\partial b}{\partial t} + \overbrace{\nabla \cdot (\rho \vec{u} b) - b \nabla \cdot (\rho \vec{u})}^{\nabla \cdot (a\vec{A}) = a(\nabla \cdot \vec{A}) + (\vec{A} \cdot \nabla) a \ : \ Vector \ \ Ident.} \\
&= \rho \frac{\partial b}{\partial t} + \overbrace{\nabla \cdot (\rho \vec{u} b) + b \frac{\partial \rho}{\partial t}}^{\nabla \cdot (\rho \vec{u}) = -\frac{\partial \rho}{\partial t} \ : \ Cons. \ \ of \ \ mass} \\
\rho \frac{Db}{Dt} &= \underbrace{\frac{\partial \rho b}{\partial t} + \nabla \cdot (\rho \vec{u} b)}_{Eulerian \ \ Conservation \ \ Form \ \ (CFD)}
\end{aligned}
$$

ASIDE: Control Volume Integral Conservation Transport Equation

- This particular result is rather interesting:

$$\begin{aligned} \rho \frac{Db}{Dt} &= \rho \frac{\partial b}{\partial t} + \rho \left(\vec{u} \cdot \nabla \right) b \\ &= \frac{\partial \rho b}{\partial t} + \nabla \cdot \left(\rho \vec{u} b \right) \end{aligned}$$

- Why? The two equations represent the change of a property b with respect to time. (1) is the traditional material derivative statement, (2) indicates a conservative quality – that the change in the material property b has contributions due to changes of the property b in the differential volume and the divergence (or flux) of the property b through the CS (fascinating – we'll see this *template* equation later in the CFD lecture!)

Control Volume Integral Conservation Transport Equation

- Back to the derivation:

$$\frac{dB}{dt} = \iiint_{MV} \rho \left(\frac{\partial b}{\partial t} + (\vec{u} \cdot \nabla) b \right) dV$$

$$= \overbrace{\iiint_{CV} \left(\frac{\partial (\rho b)}{\partial t} + \nabla \cdot (\rho \vec{u} b) \right) dV}^{From\ derivation\ on\ previous\ slide}$$

$$\frac{dB}{dt} = \iiint_{CV} \frac{\partial (\rho b)}{\partial t} + \overbrace{\iint_{CS} \rho b \left(\vec{u} \cdot \hat{n} \right) dS}^{Gauss's\ Theorem,\ \ see\ \ aside}$$

- This is a general statement of the time derivative of B with respect to t in a Material Volume (Lagrangian) in terms of an *Eulerian description*.

Control Volume Integral Conservation Transport Equation

- Aside: Gauss's theorem:

Control Volume Integral Conservation Transport Equation

- It is easier to deal with, however, if we pull the time rate of change out of the first integral (careful to apply Leibniz's Theorem correctly):

$$\frac{dB}{dt} = \frac{d}{dt}\iiint_{CV(t)} \rho b dV + \iint_{CS(t)} \rho b \left((\vec{u} - \vec{V}_{CS}) \cdot \hat{n} \right) dA$$

- This is more general than the statement you had learned about in undergraduate texts which implies that the control volume to be fixed in space – which is a special case of this expression.

Control Volume Integral Conservation Transport Equation

$$\underbrace{\frac{dB}{dt}}_{Lagrangian} = \underbrace{\frac{d}{dt}\iiint_{CV} \rho b dV}_{Rate\ of\ change\ of\ B\ inside\ CV} + \underbrace{\iint_{CS} \rho b \left((\vec{u} - \vec{V}_{CS}) \cdot \hat{n} \right) dA}_{Flux\ of\ B\ though\ CS}$$

Property	B	b
Mass	m_s	1
Linear Momentum	$m_s\vec{u}$	$\vec{u}$
Angular Momentum	$\vec{r} \times m_s\vec{u}$	$\vec{r} \times \vec{u}$
Kinetic Energy	$\frac{1}{2}m_s\vec{u}^2$	$\frac{\vec{u}^2}{2}$

Control Volume Integral Conservation Transport Equation

- Transport equation:

$$\frac{dB}{dt} = \frac{d}{dt}\iiint_{CV} \rho b dV + \iint_{CS} \rho b \left((\vec{u} - \vec{V}_{CS}) \cdot \hat{n} \right) dA$$

- Conservation of Mass, $b = 1$:

$$\frac{dMass}{dt}_{Lagrangian} = 0 = \frac{d}{dt}\iiint_{CV} \rho dV + \iint_{CS} \rho \left((\vec{u} - \vec{V}_{CS}) \cdot \hat{n} \right) dA$$

- The general transport theorem works for getting conservation of mass.

Integral Conservation of Momentum for a Control Volume

- Transport equation:

$$\frac{dB}{dt} = \frac{\partial}{\partial t} \iiint_{CV} \rho b dV + \iint_{CS} \rho b \left((\vec{u} - \vec{V}_{CS}) \cdot \hat{n} \right) dA$$

- Conservation of momentum, $b = \vec{u}$ – momentum per unit mass:

$$\begin{aligned} \frac{d\vec{M}}{dt} &= \frac{\partial}{\partial t} \iiint_{CV} \rho \vec{u} dV + \iint_{CS} \rho \vec{u} \left((\vec{u} - \vec{V}_{CS}) \cdot \hat{n} \right) dA \\ &= \iint_{CS} (-p\hat{n}) dS + \int \int_{CS} \tau dS + \iiint_{CV} \rho \vec{g} dV + \vec{F}_{applied} \end{aligned}$$

Integral Conservation of Momentum for a Control Volume

Let's pay some attention to the RHS of the equation and what it says:

$$= \iint_{CS} (-p\hat{n})dS + \iint_{CS} \tau dS + \iiint_{CV} \rho\vec{g}dV + \vec{F}_{applied}$$

The conservation of linear momentum expression says that the **rate of change of momentum** is equal to the **sum of forces applied to the control volume.** Some tips:

- Whenever a solid surface is cut by the control surface, an external force should be applied to the control volume at that point.
- If in doubt, *always* draw and account for the pressure (normal stress) and shear stress on all sides of the control surface.
- The mass and any body forces should be accounted for in the problem.

Integral Conservation of Momentum for a Control Volume

- Be careful to see what the momentum equation means.
  1. It is a vector equation – one equation for each *momentum/force* direction
  2. Tricky: The normal velocity out of the control surface may not be the same direction as the momentum flux out of that control volume – see example.
  3. The forces causing the change in momentum can be volume forces, surface forces, or point loads. The total effect of all of these forces on the CV/CS must equate to the change in momentum.

$$\frac{\partial}{\partial t}\iiint_{CV} \rho u_x dV + \iint_{CS} \rho u_x \left((\vec{u} - \vec{V}_{CS}) \cdot \hat{n} \right) dA = \sum F_{Tx} \rightarrow x_{direction}$$

$$\frac{\partial}{\partial t}\iiint_{CV} \rho u_y dV + \iint_{CS} \rho u_y \left((\vec{u} - \vec{V}_{CS}) \cdot \hat{n} \right) dA = \sum F_{Ty} \rightarrow y_{direction}$$

$$\frac{\partial}{\partial t}\iiint_{CV} \rho u_z dV + \iint_{CS} \rho u_z \left((\vec{u} - \vec{V}_{CS}) \cdot \hat{n} \right) dA = \sum F_{Tz} \rightarrow z_{direction}$$

Integral Conservation of Momentum for a Control Volume: Examples

- Conservation of Mass, $b = 1$:

$$\frac{dMass}{dt}_{Lagrangian} = 0 = \frac{d}{dt}\iiint_{CV} \rho dV + \iint_{CS} \rho \left((\vec{u} - \vec{V}_{CS}) \cdot \hat{n} \right) dA$$

- Conservation of momentum, $b = \vec{u}$ – momentum per unit mass:

$$\begin{aligned} \frac{d\vec{M}}{dt} &= \frac{\partial}{\partial t}\iiint_{CV} \rho\vec{u} dV + \iint_{CS} \rho\vec{u} \left((\vec{u} - \vec{V}_{CS}) \cdot \hat{n} \right) dA \\ &= \iint_{CS} (-p\hat{n}) dS + \int\int_{CS} \tau dS + \iiint_{CV} \rho\vec{g} dV + \vec{F}_{ap} \\ &= \sum \vec{F}_T \end{aligned}$$