22.581 Module 6: The Unsteady Bernoulli Equation

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22.581 Advanced Fluid Dynamics
Spring 2017
Weds 6:30pm

Outline

1. Unsteady Bernoulli Equation
   - From Governing Equations

General equations of motion

- Equate change in momentum per unit volume with the forces per unit volume:

$$\rho\vec{a} = \rho\frac{D\vec{u}}{Dt} = -\nabla p + \rho\vec{g} + \frac{\vec{F}_{visc}}{Vol}$$

- Expanded in three dimensions:

$$\begin{aligned}
x - direction\ :\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) &= -\frac{\partial p}{\partial x} + \rho g_x + \frac{F_{x_{visc}}}{Vol} \\
y - direction\ :\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) &= -\frac{\partial p}{\partial y} + \rho g_y + \frac{F_{y_{visc}}}{Vol} \\
z - direction\ :\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) &= -\frac{\partial p}{\partial z} + \rho g_z + \frac{F_{z_{visc}}}{Vol}
\end{aligned}$$

- IF $\frac{\vec{F}_{visc}}{Vol} = 0$, then the above equations are the Euler Equations for an incompressible fluid.

Euler Equations $\rightarrow$ Unsteady Bernoulli Equation

Assume that the flow is:

- Inviscid
- Incompressible

$$\rho\vec{a} = \rho\frac{D\vec{u}}{Dt} = -\nabla p + \rho\vec{g}$$

- Like before, we want to re-write the above equation in streamline coordinates. First we look at the *material derivative* and expand it as:

$$\frac{\partial\vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\,\vec{u} = \frac{\partial\vec{u}}{\partial t} + \nabla\left(\frac{|u|^2}{2}\right) - \vec{u}\times(\nabla\times\vec{u})$$

Euler Equations $\rightarrow$ Unsteady Bernoulli Equation

$$\rho \left[\frac{\partial \vec{u}}{\partial t} + \nabla \left(\frac{|u|^2}{2} \right) - \vec{u} \times (\nabla \times \vec{u}) \right] = -\nabla p + \rho \vec{g}$$

- We then integrate from point *1* to *2* along a streamline:

$$\int_1^2 \left[\frac{\partial \vec{u}}{\partial t} + \nabla \left(\frac{u^2}{2} \right) - \vec{u} \times (\nabla \times \vec{u}) = -\frac{\nabla p}{\rho} - \nabla (gz) \right] d\vec{s}$$

- Here $d\vec{s}$ is a differential length of the streamline.

Euler Equations → Unsteady Bernoulli Equation

$$\int_1^2 \left[\frac{\partial \vec{u}}{\partial t} + \nabla\left(\frac{u^2}{2}\right) - \vec{u} \times (\nabla \times \vec{u}) = -\frac{\nabla p}{\rho} - \nabla(gz)\right] d\vec{s}$$

- Let's expand and rearrange terms a little:

$$\begin{aligned}\int_1^2 \vec{u} \times (\nabla \times \vec{u}) \cdot d\vec{s} \quad &= \quad \int_1^2 \frac{\partial \vec{u}}{\partial t} \cdot d\vec{s} + \int_1^2 \underbrace{\nabla\left(\frac{u^2}{2}\right) \cdot d\vec{s}}_{d\left(\frac{V^2}{2}\right)} \\ &+ \quad \int_1^2 \underbrace{\frac{\nabla p}{\rho} \cdot d\vec{s}}_{\frac{dp}{\rho}} + \int_1^2 \underbrace{\nabla gz \cdot d\vec{s}}_{d(gz)}\end{aligned}$$

Euler Equations $\rightarrow$ Unsteady Bernoulli Equation

- Which simplifies to:

$$\int_1^2 \underbrace{\vec{u} \times (\nabla \times \vec{u}) \cdot d\vec{s}}_{\textit{see comments below}} = \int_1^2 \frac{\partial \vec{u}}{\partial t} \cdot d\vec{s} + \int_1^2 \frac{dp}{\rho} + \left[\frac{\vec{u}_2^2}{2} - \frac{\vec{u}_1^2}{2} \right] + [gz_2 - gz_1]$$

- Let's examine the first term of the above equation. The integrand is zero along the streamline (the dot product of a vector with a orthonormal vector is zero, by definition):

$$\int_1^2 \vec{u} \times (\nabla \times \vec{u}) \cdot d\vec{s} = 0 \quad \textit{along streamline}$$

- If the flow is *irrotational* everywhere (ie. $\nabla \times \vec{u} = 0$) then the above restriction is not just along a streamline anymore.

Euler Equations $\rightarrow$ Unsteady Bernoulli Equation

- If we also set $\rho = const.$ then:

$$\int_1^2 \frac{\partial \vec{u}}{\partial t} \cdot d\vec{s} + \frac{p_2}{\rho} + \frac{u_2^2}{2} + gz_2 = \frac{p_1}{\rho} + \frac{u_1^2}{2} + gz_1$$

- True...
  1. between any two points along a streamline
  2. everywhere if $\nabla \times \vec{u} = 0$
- If $\nabla \times \vec{u} = 0$ we can also write $\nabla \phi = \vec{u}$, and then $\int_1^2 \frac{\partial \vec{u}}{\partial t} \cdot d\vec{s} = \frac{\partial \phi}{\partial t}$
- The assumptions are:
  1. Flow is inviscid
  2. Flow is incompressible