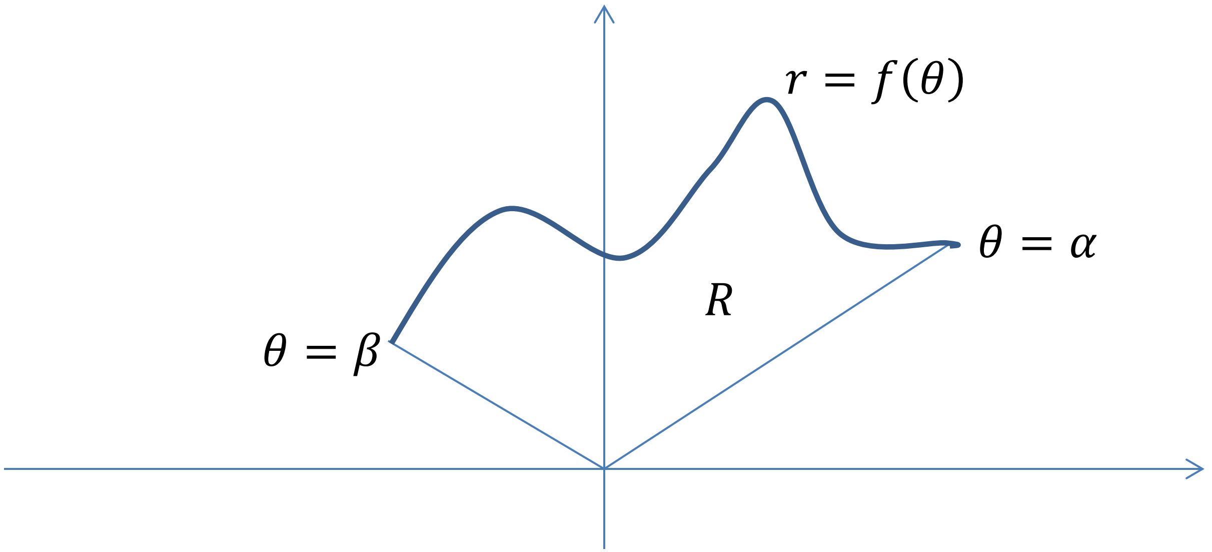
Areas and Lengths in Polar Coordinates

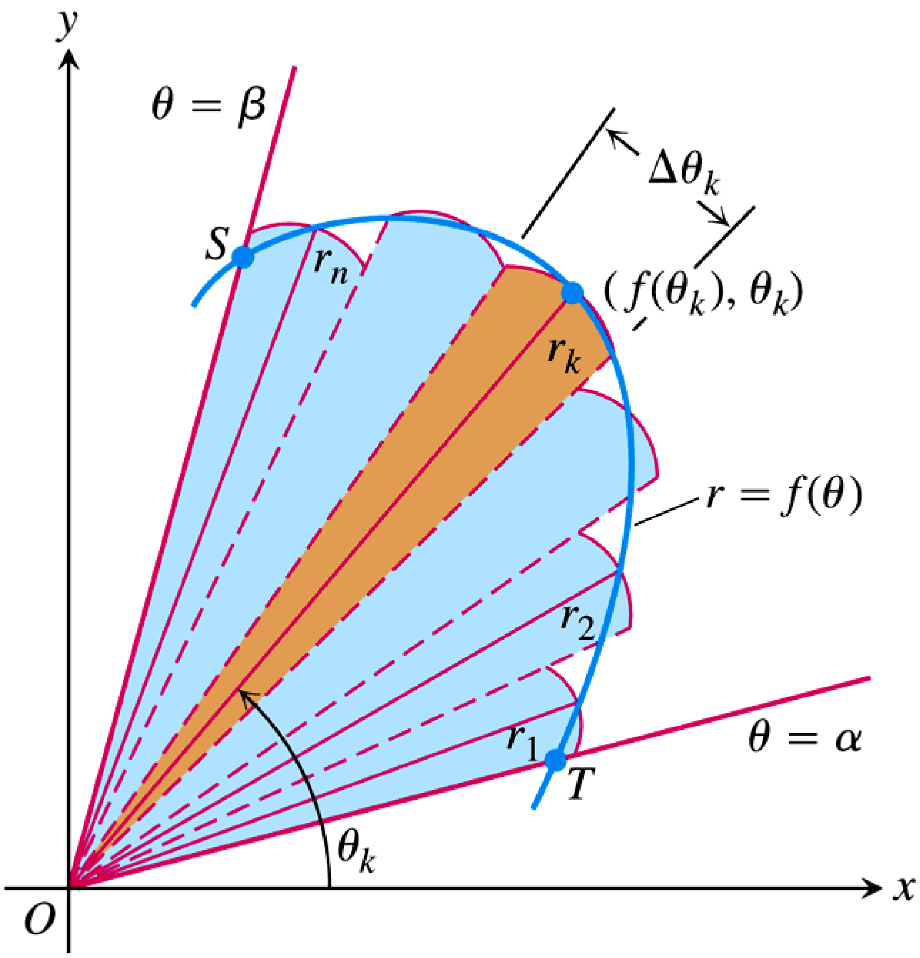
Part 1: Areas

Problem

Find the area of the region R between a polar curve $r = f(\theta)$ and two lines, $\theta = \alpha$ and $\theta = \beta$.



Subdivide Area into Subregions



Estimate Area of Subregions

If $\Delta\theta_k$ is not too large, we can approximate the area A_k by the area of a *sector* having central angle $\Delta\theta_k$ and radius $r_k = f(\theta_k)$.

$$\begin{aligned} A_k &\approx \text{area of sector} \\ &= \frac{1}{2} r_k^2 \Delta\theta_k \\ &= \frac{1}{2}\big(f(\theta_k)\big)^2 \Delta\theta_k \end{aligned}$$

Add Up Areas of Subregions

$$A = A_1 + A_2 + A_3 + \cdots + A_n$$

$$= \sum_{k=1}^{n} A_k$$

$$\approx \sum_{k=1}^{n} \frac{1}{2}\left(f(\theta_k)\right)^2 \Delta\theta_k$$

Increase Number of Subdivisions

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2}\left(f(\theta_k)\right)^2 \Delta\theta_k$$

$$= \int_{\alpha}^{\beta} \frac{1}{2}\left(f(\theta)\right)^2 d\theta$$

Area Enclosed by a Polar Curve

If $f(\theta)$ is continuous and nonnegative for $\alpha \leq \theta \leq \beta$, then the area A enclosed by the polar curve $r = f(\theta)$ and the lines $\theta = \alpha$ and $\theta = \beta$ is

$$A = \int_{\alpha}^{\beta} \frac{1}{2}\big(f(\theta)\big)^2 \, d\theta$$

or equivalently

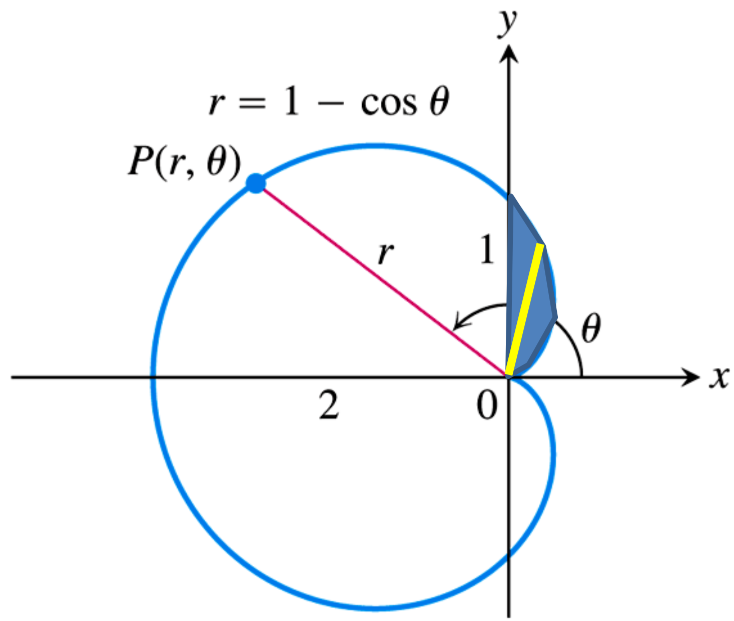
$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

Steps to Calculating Area

1. Sketch the region R whose area is to be determined.
2. Draw an arbitrary "radial line" from the origin to the boundary curve $r = f(\theta)$.
3. Ask, "Over what interval of values must θ vary in order for the radial line to sweep out the region R?"
4. Your answer in Step 3 will determine the lower and upper limits of integration.

Example 1

Find the area of the region in the first quadrant within the cardioid $r = 1 - \cos\theta$.



Solution:

The region is colored in blue and a typical radial line is shown in yellow.

Example 1 (continued)

For the radial line to sweep out the region, θ must vary from 0 to $\frac{\pi}{2}$.

$$A = \int_0^{\frac{\pi}{2}} \frac{1}{2} r^2 \, d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2} (1 - \cos\theta)^2 \, d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} (1 - 2\cos\theta + \cos^2\theta) \, d\theta = \cdots = \frac{3}{8}\pi - 1$$

Example 2

Find the entire area within the cardioid of Example 1.

Solution:

For the radial line to sweep out the entire cardioid, θ must vary from 0 to 2π.

$$A = \int_0^{2\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{2\pi} \frac{1}{2} (1 - \cos\theta)^2 \, d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta) \, d\theta = \cdots = \frac{3}{2}\pi$$

Example 3

Find the area of the region that is outside the cardioid $r = 1 - \cos\theta$ and inside the circle $r = 1$.

<u>Solution</u>:

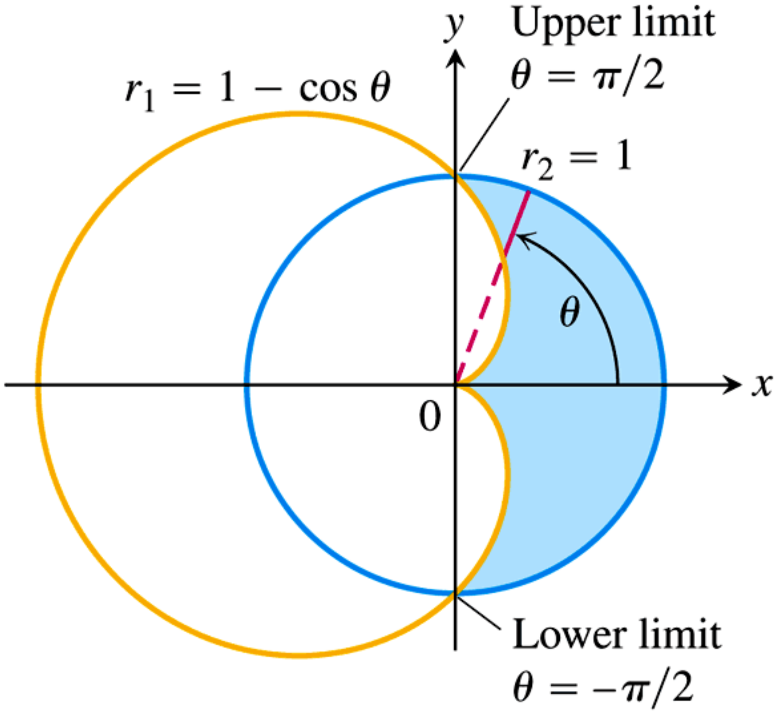
To sketch the region, we need to know where the circle and cardioid intersect. To find these points, we equate the given expressions for r.

Example 3 (continued)

$$1 - \cos\theta = 1$$
$$0 = \cos\theta$$

or

$$\theta = -\frac{\pi}{2} \text{ and } \theta = \frac{\pi}{2}.$$



The desired area can be obtained by subtracting the area of the cardioid in Quadrants I and IV from the area of the circle in Quadrants I and IV.

Example 3 (continued)

$$A = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(1)^2\, d\theta - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(1 - \cos\theta)^2\, d\theta$$

$$= \frac{\pi}{2} - \left(\frac{3\pi}{4} - 2\right)$$

$$= 2 - \frac{\pi}{4}$$

http://www.pleacher.com/mp/mhumor/comx/calculus/polar.jpg