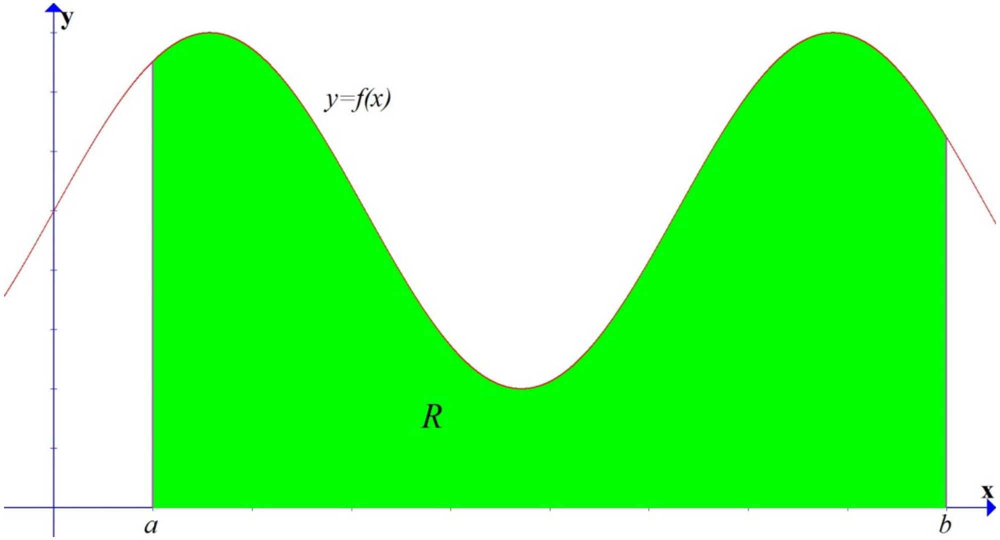
Area and Estimating with Finite Sums

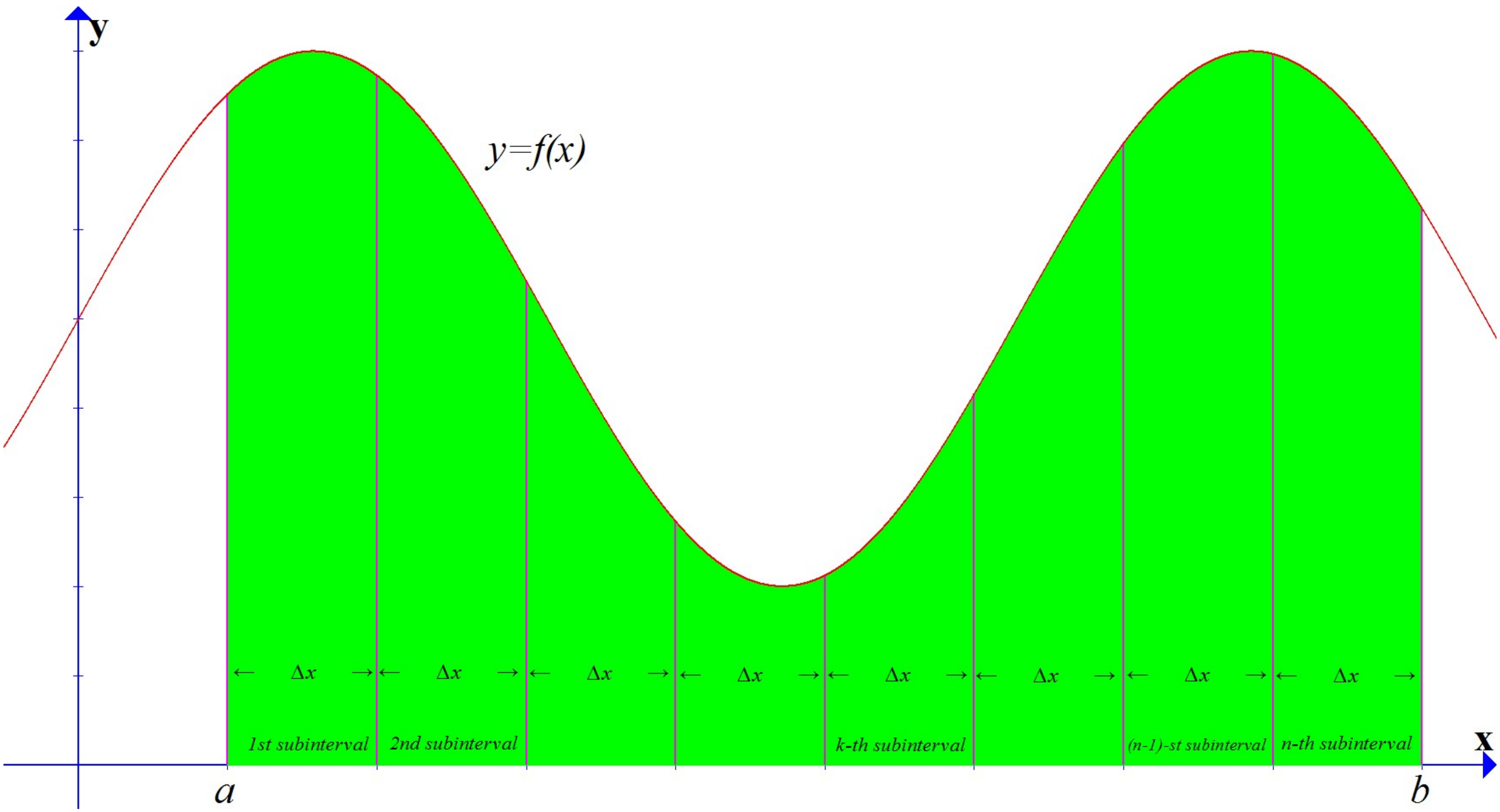
Part 1

Problem:

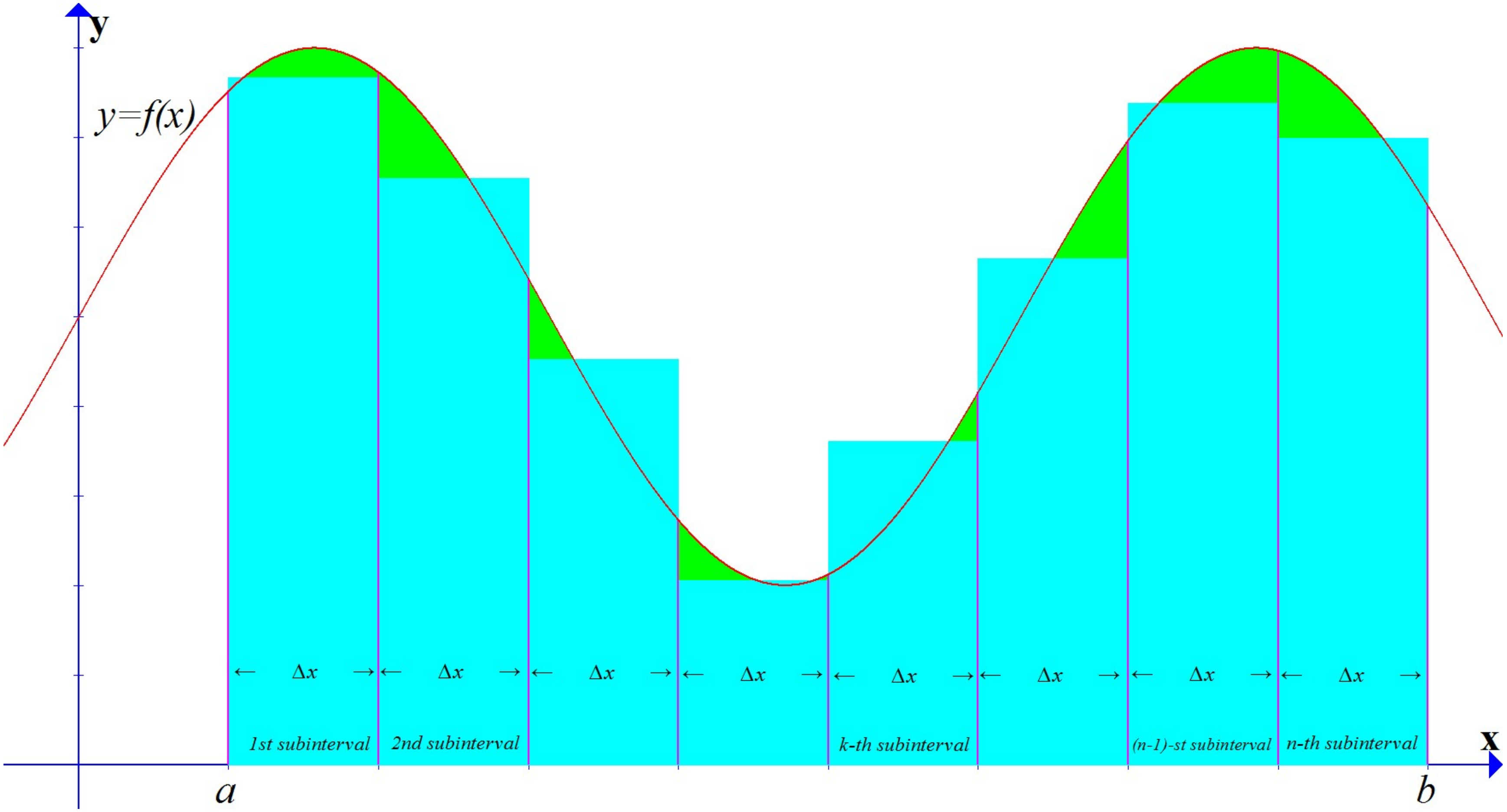
Find the area of a region R bounded below by the x-axis, on the sides by the lines $x = a$ and $x = b$, and above by a curve $y = f(x)$, where f is continuous on $[a, b]$ and $f(x) \geq 0$ for all x in $[a, b]$



First Step: Subdivide $[a, b]$ into n subintervals of equal width of Δx

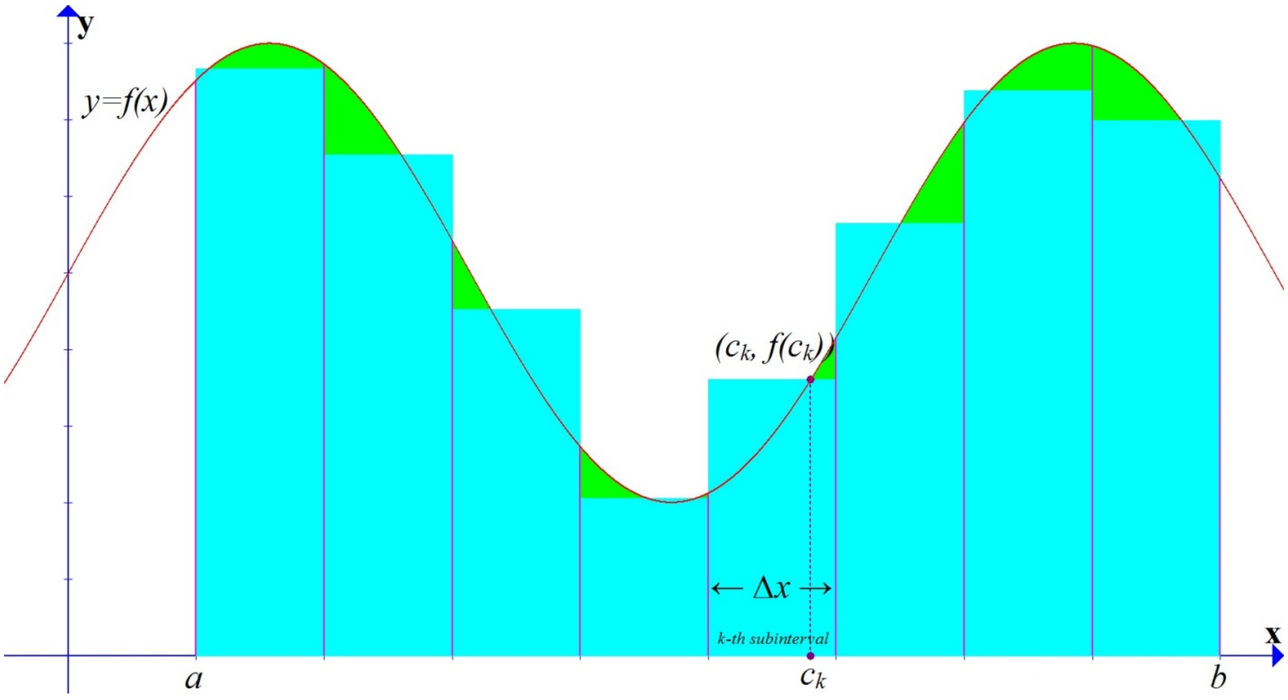


Second Step: On each subinterval, draw a rectangle to approximate the area of the curve over the subinterval



Third Step: Determine the area of each rectangle

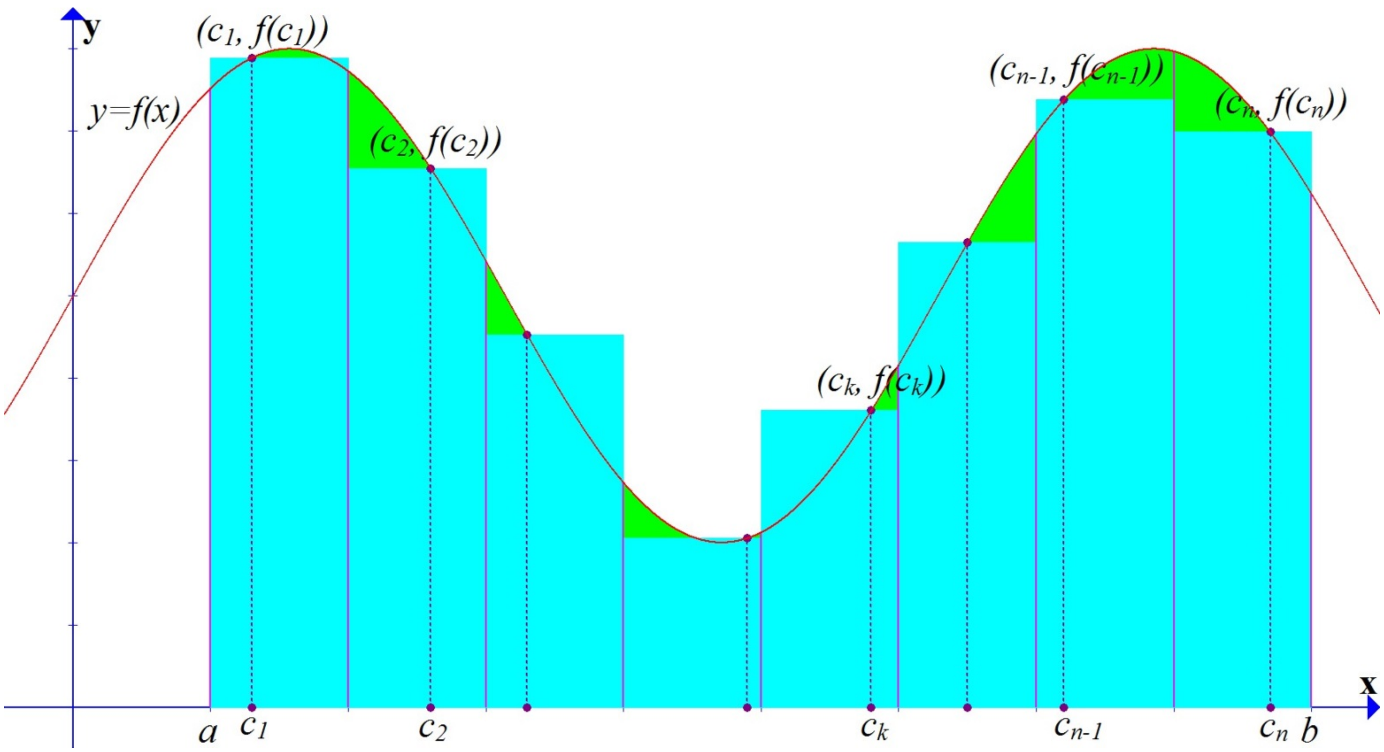
- Width of the *k*-th rectangle = Δx
- Height of *k*-th rectangle = $f(c_k)$
- Area of *k*-th rectangle = $A_k = f(c_k) \cdot \Delta x$



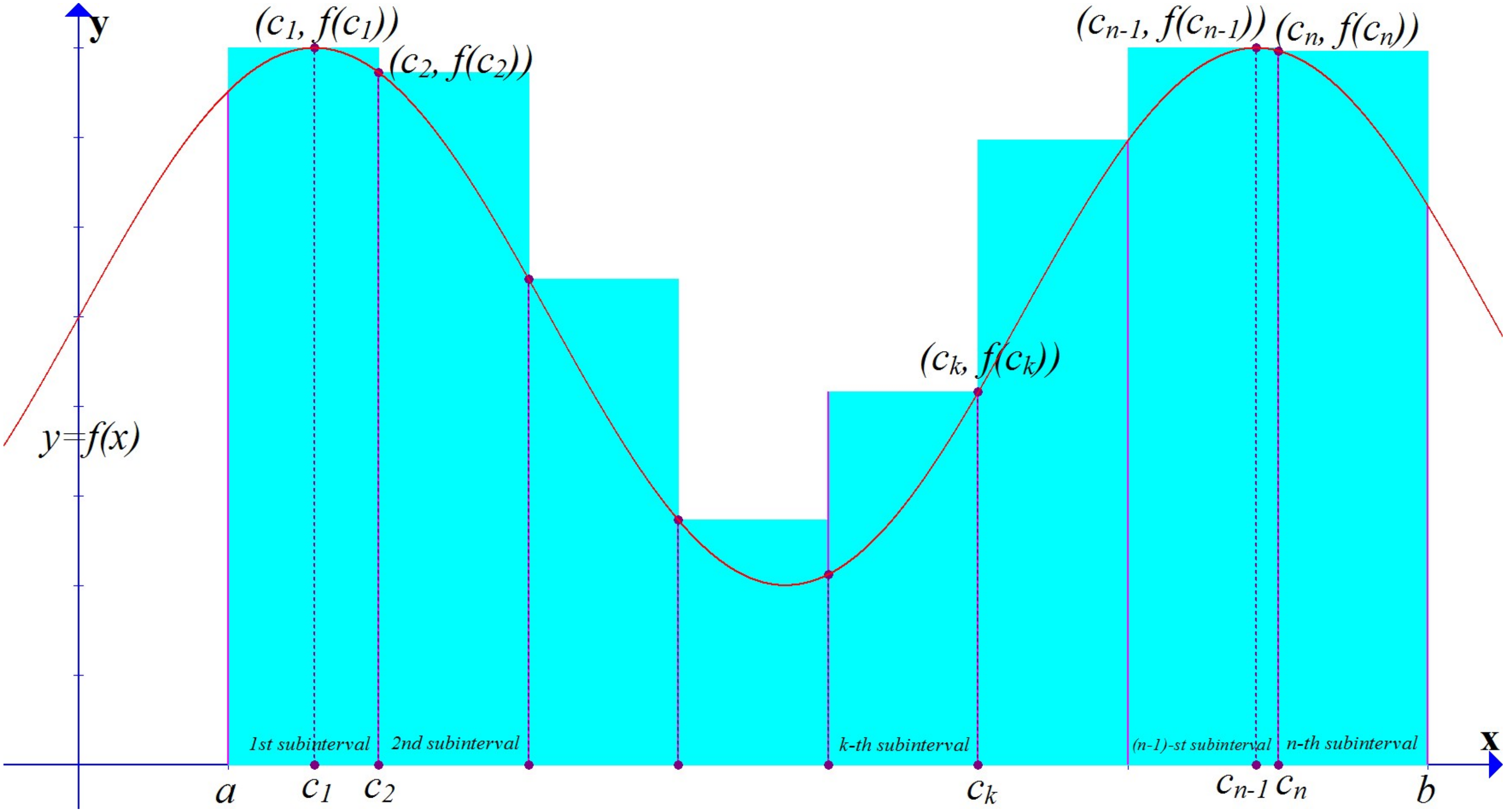
Fourth Step: Sum up the areas of the rectangles to approximate the area under the curve

$$A \approx A_1 + A_2 + A_3 + \cdots + A_n$$

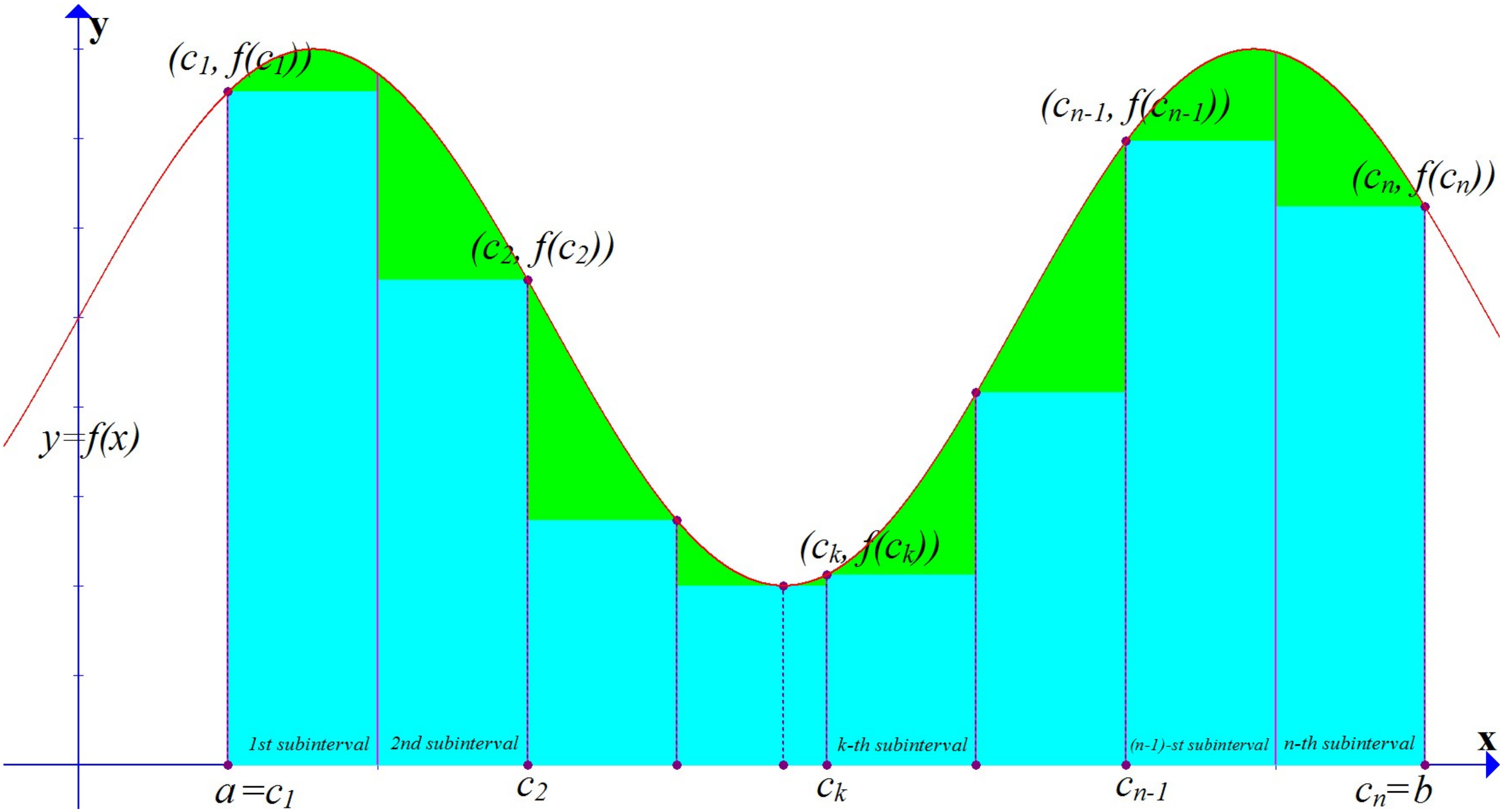
$$= f(c_1) \cdot \Delta x + f(c_2) \cdot \Delta x + f(c_3) \cdot \Delta x + \cdots + f(c_n) \cdot \Delta x$$



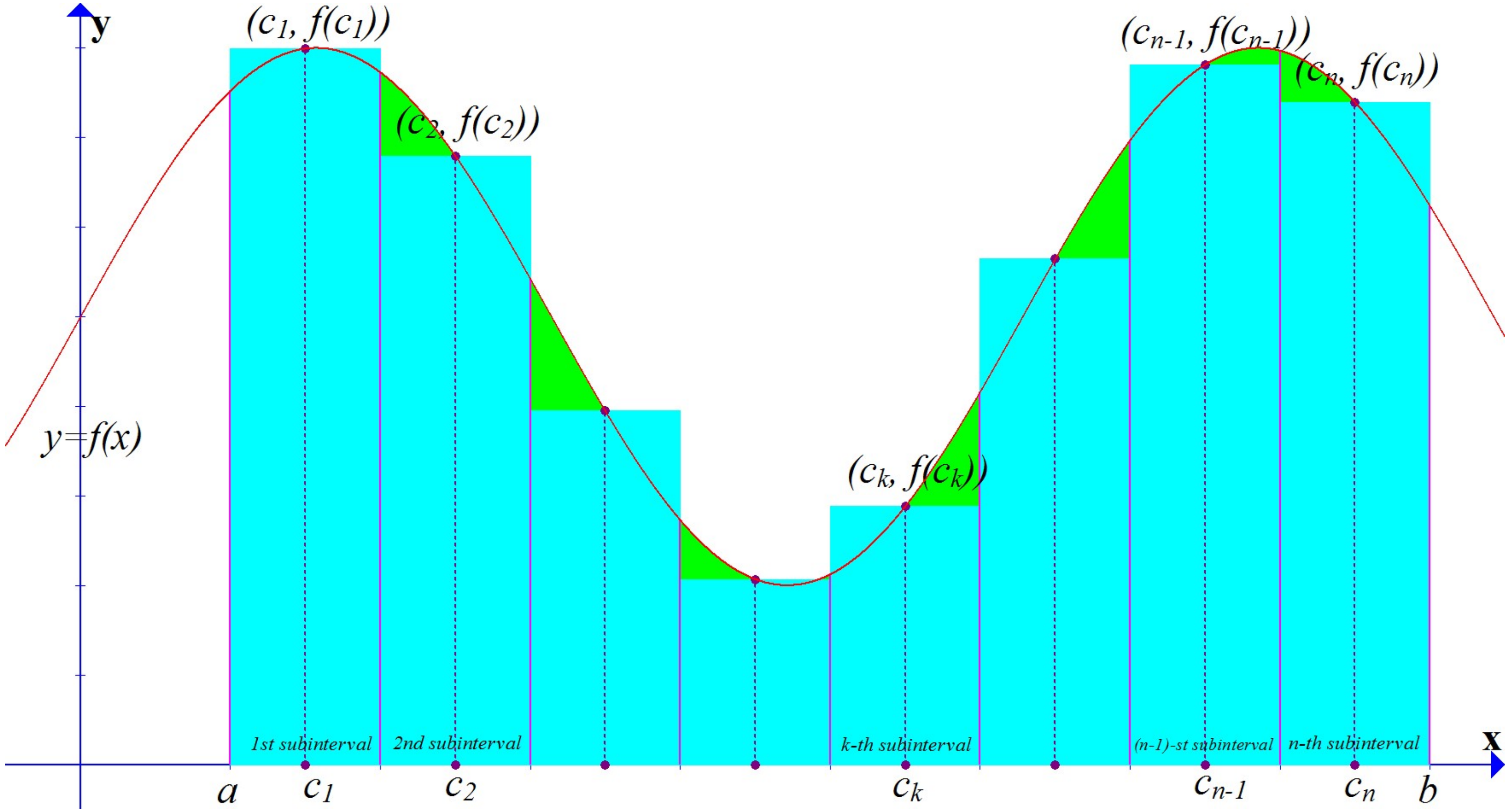
Upper sum method: $(c_k, f(c_k))$ is the absolute maximum of $y = f(x)$ on the k-th subinterval



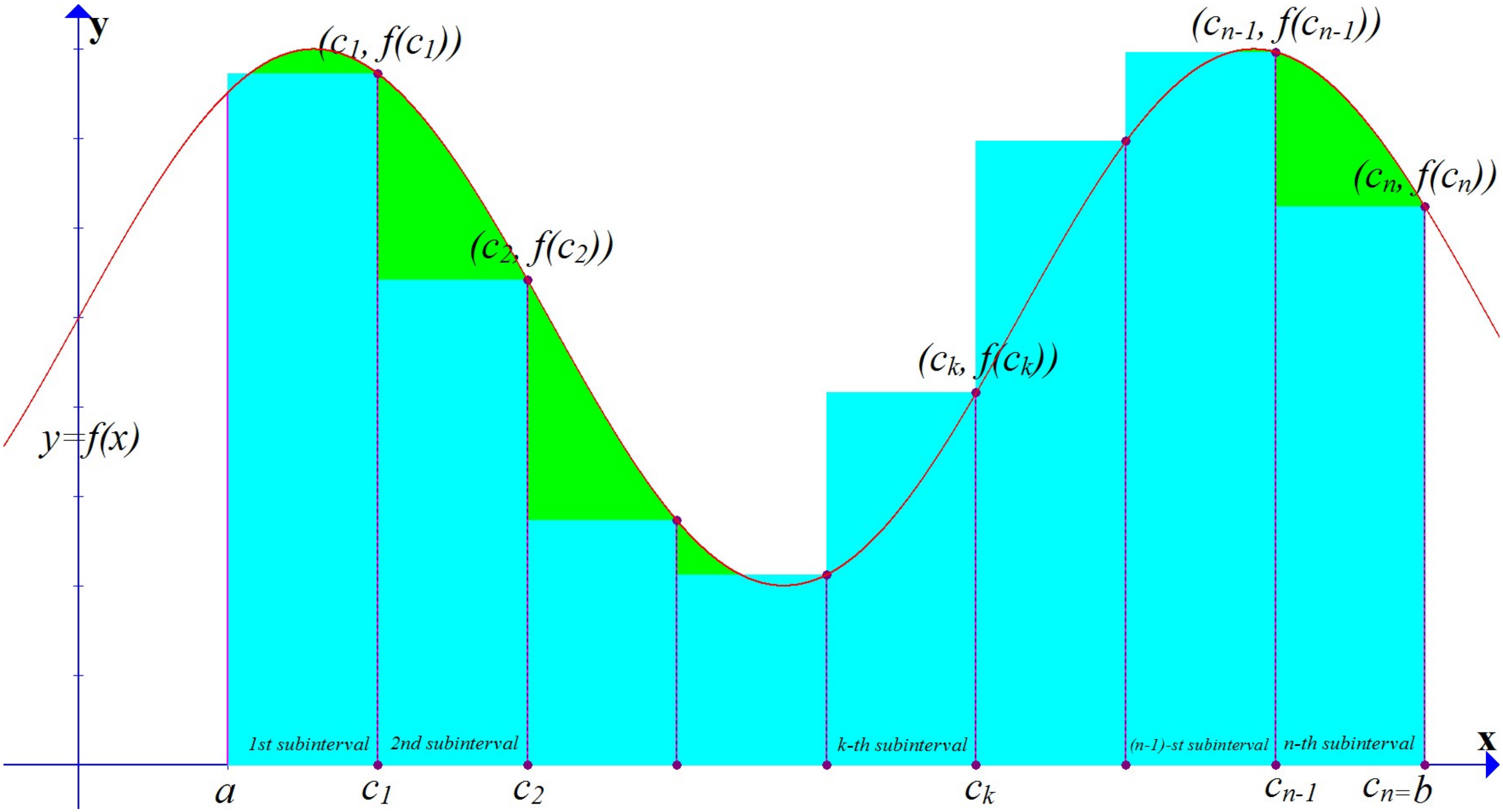
Lower sum method: $(c_k, f(c_k))$ is the absolute minimum of $y = f(x)$ on the k-th subinterval



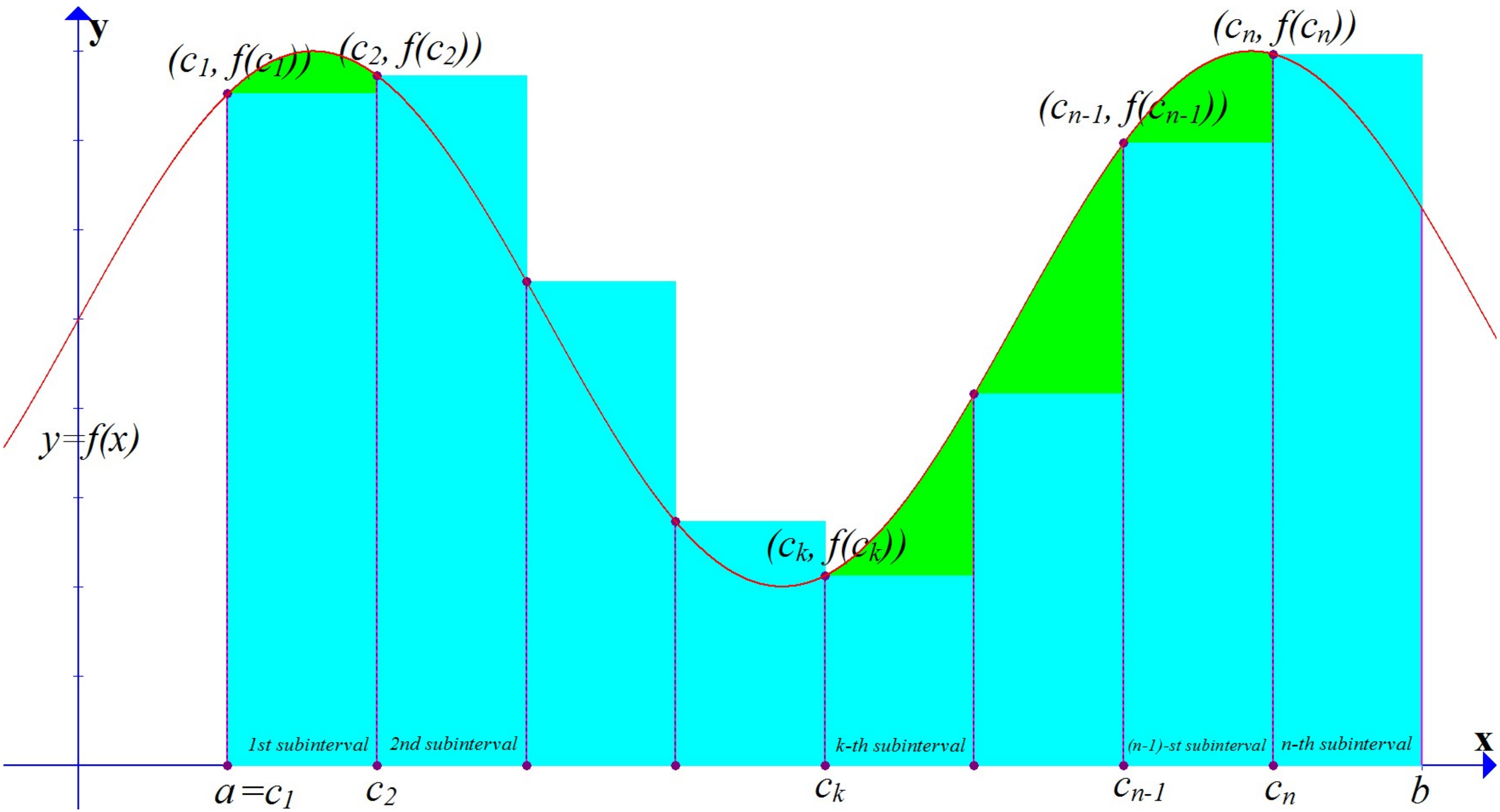
Midpoint method: c_k is the midpoint of the k-th subinterval



Right endpoint method: c_k is the right endpoint of the k-th subinterval



Left endpoint method: c_k is the left endpoint of the k-th subinterval



Example 1

Use finite approximation to estimate the area under $f(x) = 3x + 1$ over $[2,6]$ using a lower sum with four rectangles of equal width.

<u>Solution:</u>

$$f(x) = 3x + 1$$

$$a = 2 \text{ and } b = 6$$

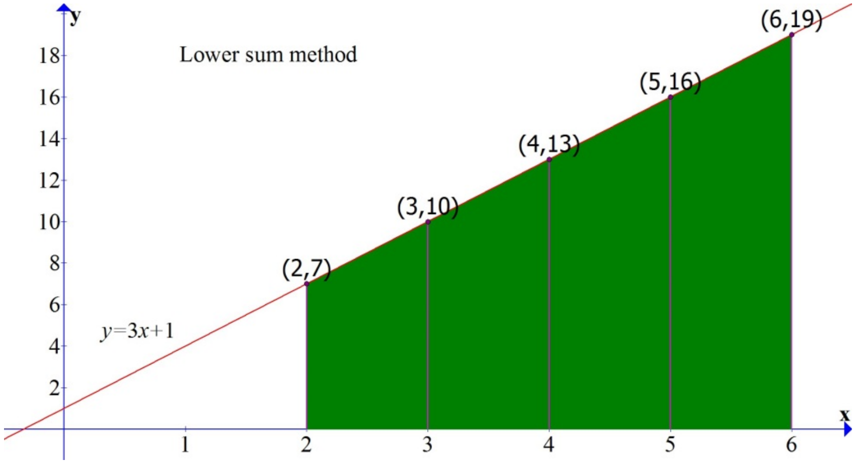
$$n = 4$$

Example 1 (continued)
$f(x) = 3x + 1$ over $[2,6]$

$$\Delta x = \frac{\text{width of the interval}[a, b]}{\text{number of subintervals}}$$

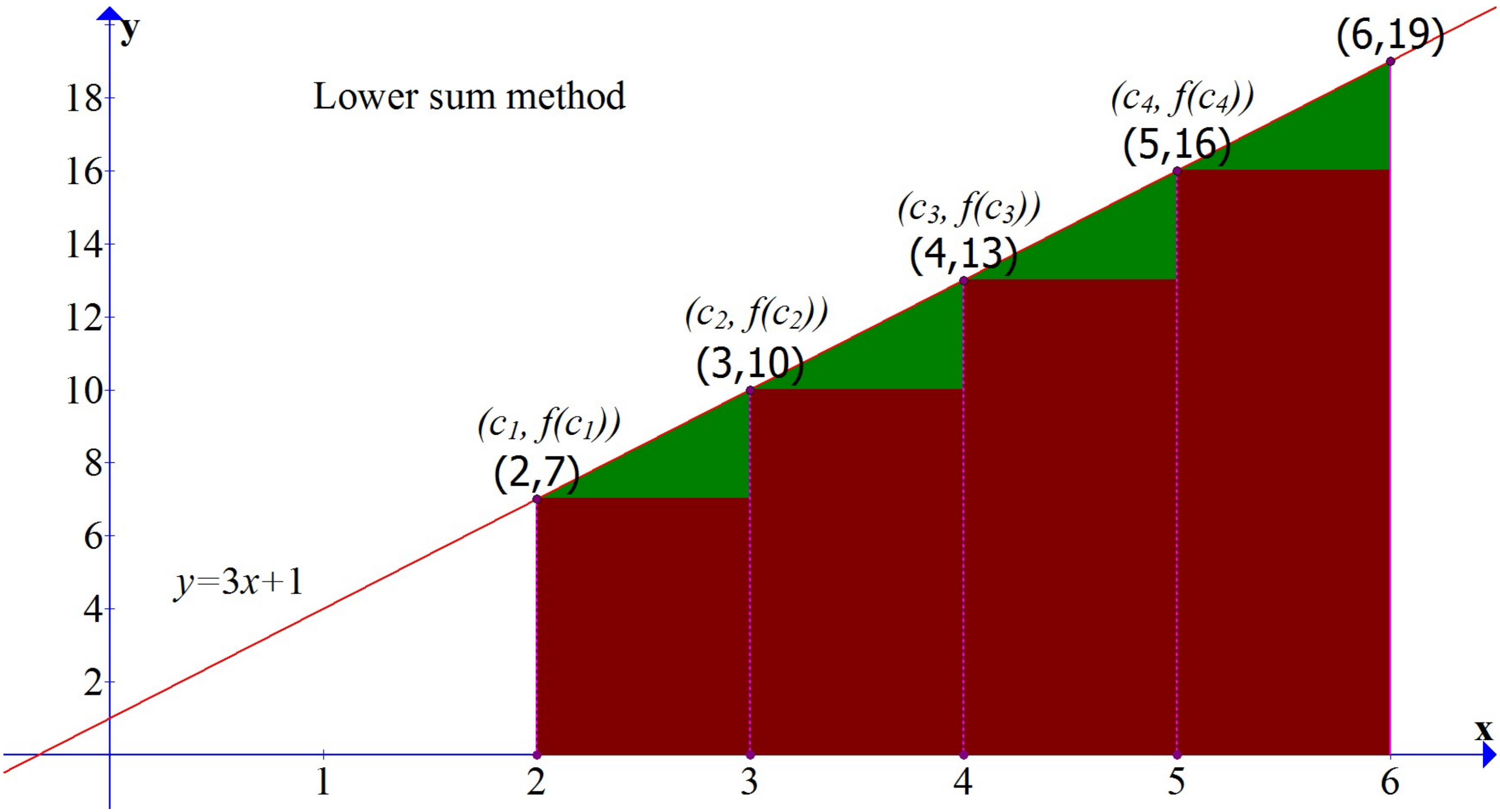
$$= \frac{b - a}{n}$$

$$\Delta x = \frac{6 - 2}{4} = 1$$



Note: in this example, the lower sum method is the same as the left endpoint method

Example 1 (continued)
$f(x) = 3x + 1$ over [2,6]

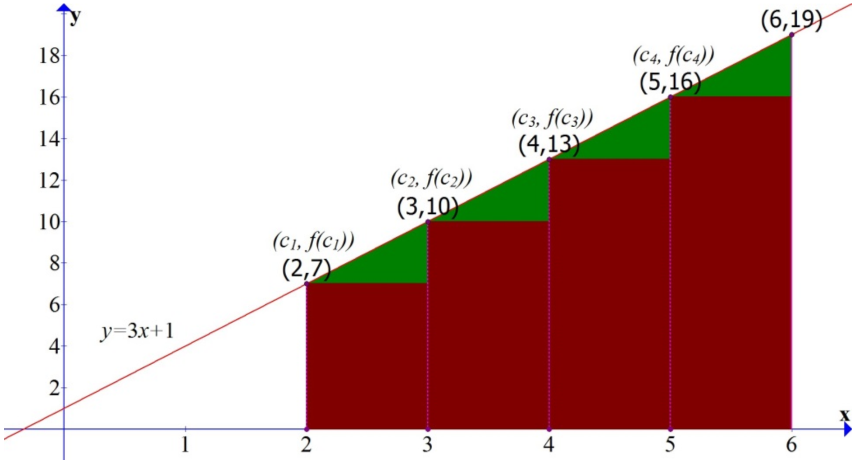


Example 1 (continued)
$f(x) = 3x + 1$ over $[2,6]$

$$A \approx f(c_1) \cdot \Delta x + f(c_2) \cdot \Delta x + f(c_3) \cdot \Delta x + f(c_4) \cdot \Delta x$$

$$A \approx f(2) \cdot 1 + f(3) \cdot 1 + f(4) \cdot 1 + f(5) \cdot 1$$

$$A \approx 7 + 10 + 13 + 16 = 46$$



Answer: The area under the curve $y = 3x + 1$ over the interval $[2,6]$ is approximately 46.

Example 2

Using the midpoint rule, estimate the area under $f(x) = \frac{1}{x}$ over $[1,9]$ using four rectangles.

Solution:

$$f(x) = \frac{1}{x}$$

$$a = 1 \text{ and } b = 9$$

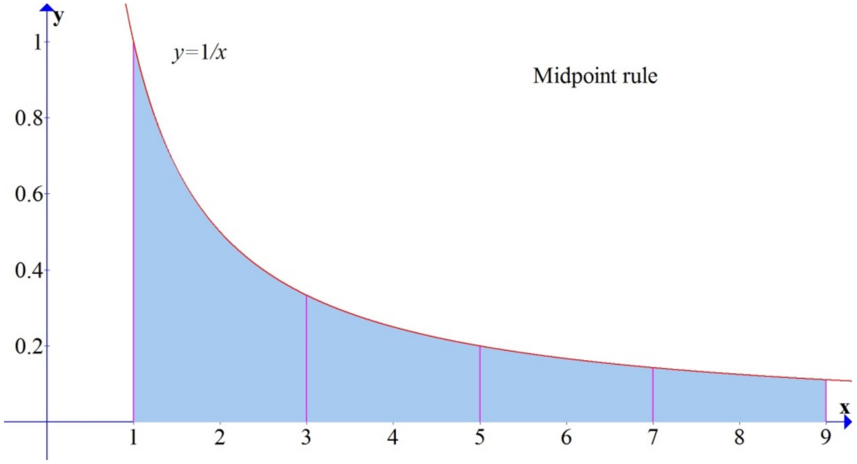
$$n = 4$$

Example 2 (continued)
$f(x) = \frac{1}{x}$ over $[1,9]$

$$\Delta x = \frac{\text{width of the interval}[a, b]}{\text{number of subintervals}}$$

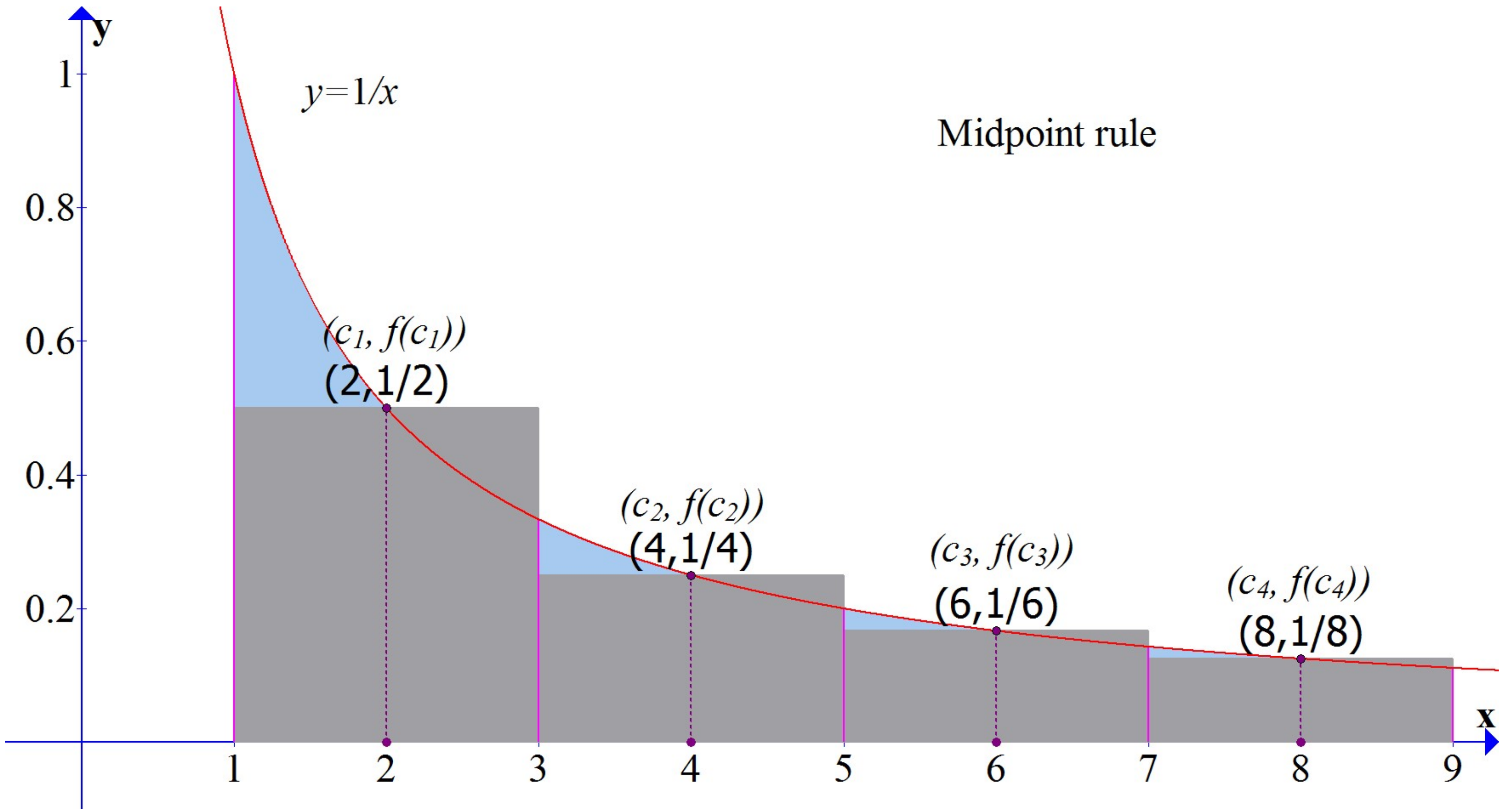
$$= \frac{b-a}{n}$$

$$\Delta x = \frac{9-1}{4} = 2$$



Example 2 (continued)

$$f(x) = \frac{1}{x} \text{ over } [1,9]$$

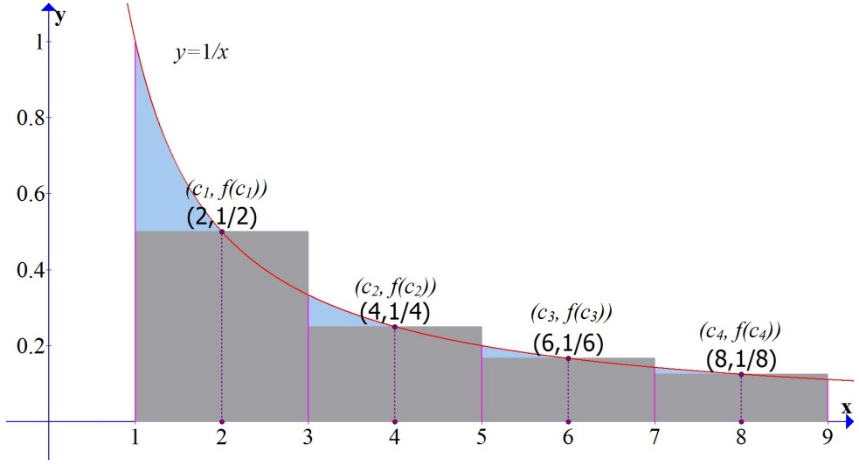


Example 2 (continued)
$f(x) = \frac{1}{x}$ over $[1,9]$

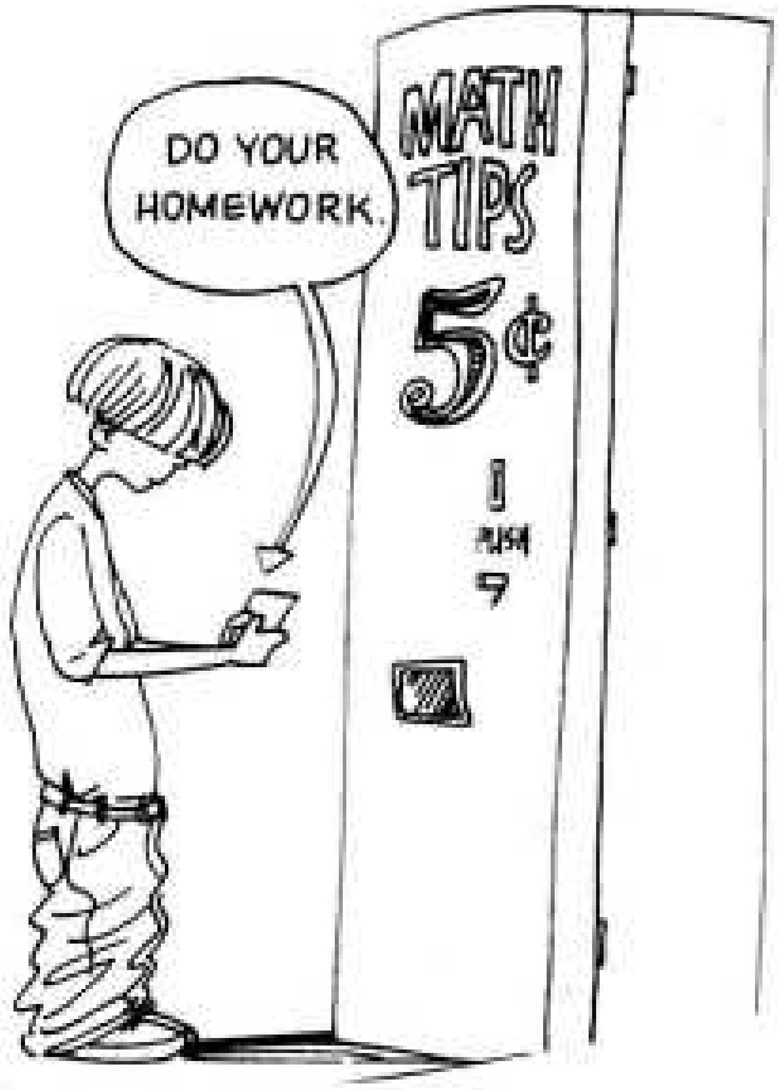
$$A \approx f(c_1) \cdot \Delta x + f(c_2) \cdot \Delta x + f(c_3) \cdot \Delta x + f(c_4) \cdot \Delta x$$

$$A \approx f(2) \cdot 2 + f(4) \cdot 2 + f(6) \cdot 2 + f(8) \cdot 2$$

$$A \approx \frac{1}{2} \cdot 2 + \frac{1}{4} \cdot 2 + \frac{1}{6} \cdot 2 + \frac{1}{8} \cdot 2 = \frac{25}{12}$$



Answer: The area under the curve $y = \frac{1}{x}$ over the interval $[1,9]$ is approximately $\frac{25}{12}$.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