Sigma Notation and Limits of Finite Sums

Part 1: Sigma Notation

Sigma Notation

$$\sum_{k=a}^{b} f(k) = f(a) + f(a+1) + f(a+2) + \cdots + f(b-1) + f(b)$$

- Σ is the Greek letter capital sigma
- k is the **index of summation**
- a is the **lower limit of summation**
- b is the **upper limit of summation**

Examples of Sigma Notation

$$\sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2$$

$$\sum_{k=4}^{8} k^3 = 4^3 + 5^3 + 6^3 + 7^3 + 8^3$$

$$\sum_{k=1}^{5} 2k = 2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3 + 2 \cdot 4 + 2 \cdot 5$$

$$= 2 + 4 + 6 + 8 + 10$$

Examples of Sigma Notation (continued)

$$\sum_{k=0}^{5}(2k+1) = 1+3+5+7+9+11$$

$$\sum_{k=0}^{5}(-1)^k(2k+1) = 1-3+5-7+9-11$$

$$\sum_{k=-3}^{1} k^3 = (-3)^3+(-2)^3+(-1)^3+0^3+1^3$$

$$\sum_{k=1}^{3} k\sin\left(\frac{k\pi}{5}\right) = \sin\left(\frac{\pi}{5}\right)+2\sin\left(\frac{2\pi}{5}\right)+3\sin\left(\frac{3\pi}{5}\right)$$

Examples of Sigma Notation (continued)

$$\sum_{k=2}^{2} k^3 = 2^3$$

$$\sum_{k=4}^{8} 2 = 2 + 2 + 2 + 2 + 2 = 10$$

$$\sum_{k=1}^{5} 2k = \sum_{k=0}^{4} (2k + 2) = \sum_{k=2}^{6} (2k - 2) = 2 + 4 + 6 + 8 + 10$$

Example 1

Express

$$\sum_{k=3}^{7} 5^{k-2}$$

in sigma notation so that the lower limit of summation is 0 rather than 3.

Example 1 (continued)

$$\sum_{k=3}^{7} 5^{k-2}$$

<u>Solution</u>:

Let $j = k - 3 \Rightarrow k = j + 3$. Then we have:

k=3	j=0
k=4	j=1
k=5	j=2
k=6	j=3
k=7	j=4

So, as k goes from 3 to 7, j goes from 0 to 4.

Example 1 (continued)

$$\sum_{k=3}^{7} 5^{k-2}$$

$$\sum_{k=3}^{7} 5^{k-2} = \sum_{j=0}^{4} 5^{(j+3)-2} = \sum_{j=0}^{4} 5^{j+1} = \sum_{k=0}^{4} 5^{k+1}$$

<u>Answer</u>:

$$\sum_{k=3}^{7} 5^{k-2} = \sum_{k=0}^{4} 5^{k+1}$$

General Sum

To represent a general sum, use letters with subscripts:

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$$

Algebraic Properties of Sigma Notation

$$\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

$$\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$$

$$\sum_{k=1}^{n} c \cdot a_k = c \sum_{k=1}^{n} a_k$$

$$\sum_{k=1}^{n} c = n \cdot c$$

Important formulas

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + (n-1) + n = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + (n-1)^3 + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Example 2

Evaluate

$$\sum_{k=1}^{30} k(k+1)$$

<u>Solution</u>:

Using $\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$ we get:

$$\sum_{k=1}^{30} k(k+1) = \sum_{k=1}^{30}(k^2 + k) = \sum_{k=1}^{30} k^2 + \sum_{k=1}^{30} k$$

Example 2 (continued)

$$\sum_{k=1}^{30} k(k+1)$$

Next, using $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ and $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ we get:

$$\sum_{k=1}^{30} k^2 + \sum_{k=1}^{30} k = \frac{30(30+1)(2(30)+1)}{6} + \frac{30(30+1)}{2} = 9920$$

<u>Answer</u>:

$$\sum_{k=1}^{30} k(k+1) = 9920$$

$\sqrt{-1} \quad 2^3 \quad \Sigma \quad \pi$

and it was delicious