Sigma Notation and Limits of Finite Sums

Part 2: Limits of Finite Sums and Riemann Sums

Limits of Finite Sums

Recall:

If f is a non-negative, continuous function on the interval $[a, b]$ and if A is the area under the curve $y = f(x)$ over the interval $[a, b]$, then

$$A \approx f(c_1)\Delta x + f(c_2)\Delta x + f(c_3)\Delta x + \cdots + f(c_n)\Delta x$$

where $[a, b]$ has been subdivided into n subintervals of equal width of Δx and c_k is some point in the k-th subinterval.

Limits of Finite Sums

- We can now write this approximation in sigma notation:

$$A \approx \sum_{k=1}^{n} f(c_k) \cdot \Delta x$$

- As we increase the number of subdivisions of $[a, b]$ (that is, as we increase n), this finite sum becomes more accurate. Hence, it makes sense to write

$$A = \lim_{n \to \infty} \sum_{k=1}^{n} f(c_k) \cdot \Delta x$$

Riemann Sums

While rectangles of equal widths are convenient, they are not necessary. All we need is that the widths of all rectangles decrease to zero as the number of rectangles increases to infinity.

Riemann Sums

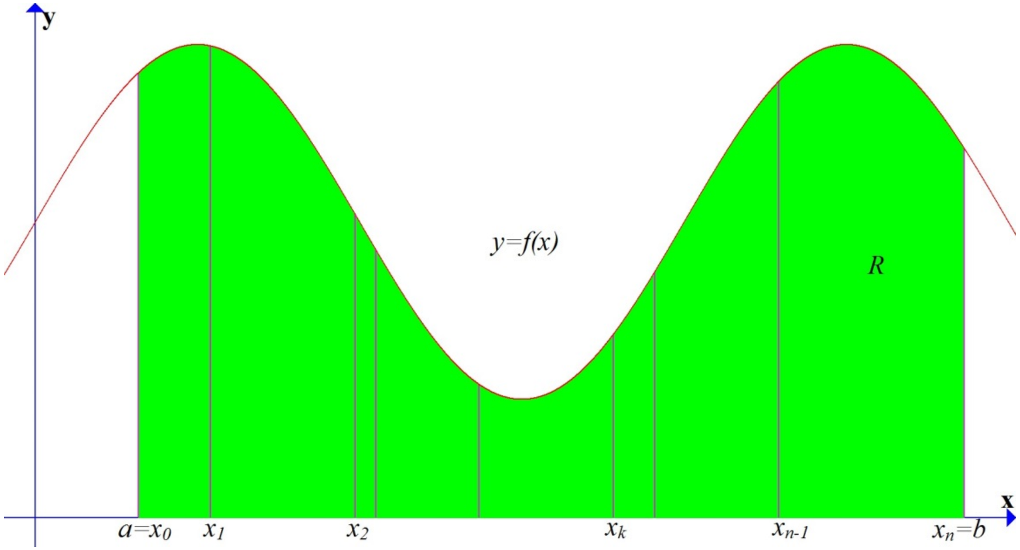
Suppose that $f(x)$ is a bounded function defined on a closed interval $[a, b]$ and that the interval $[a, b]$ is divided into n subintervals by choosing arbitrary points $x_1, x_2, x_3, \cdots, x_{n-1}$ satisfying

$$a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b$$

The set

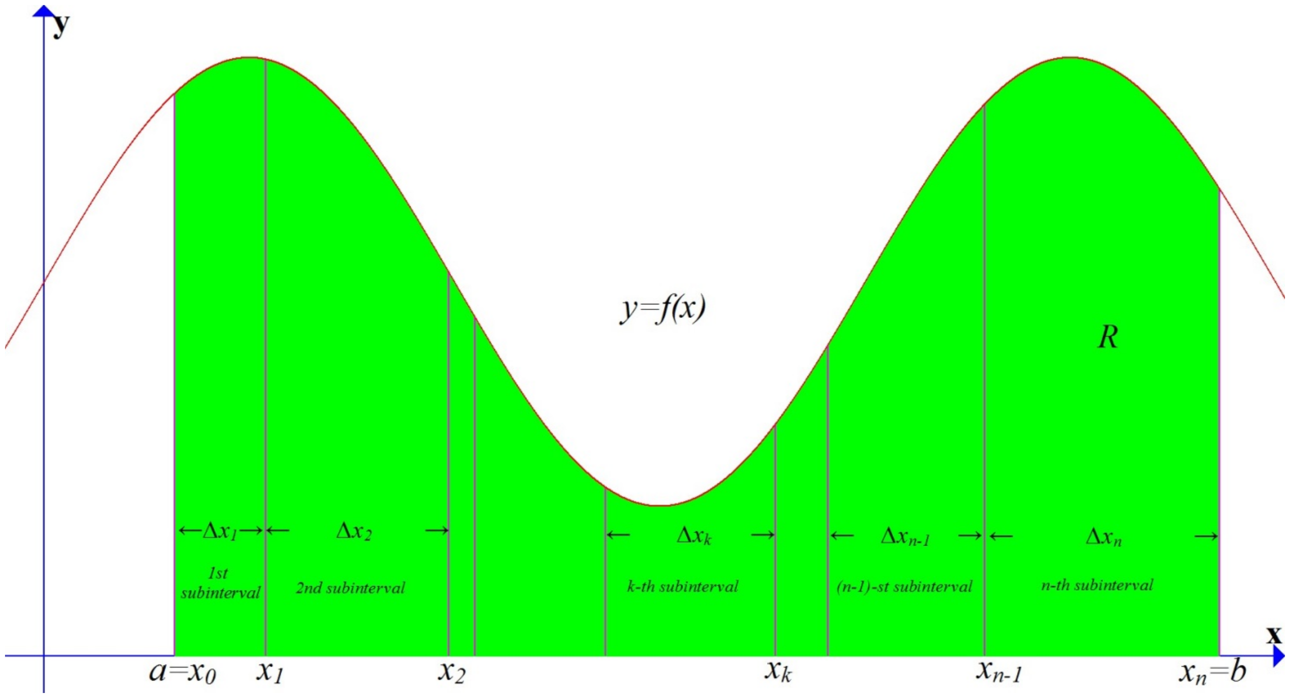
$$P = \{x_0, x_1, x_2, x_3, \cdots, x_{n-1}, x_n\}$$

is called a **partition** of $[a, b]$.



Riemann Sums

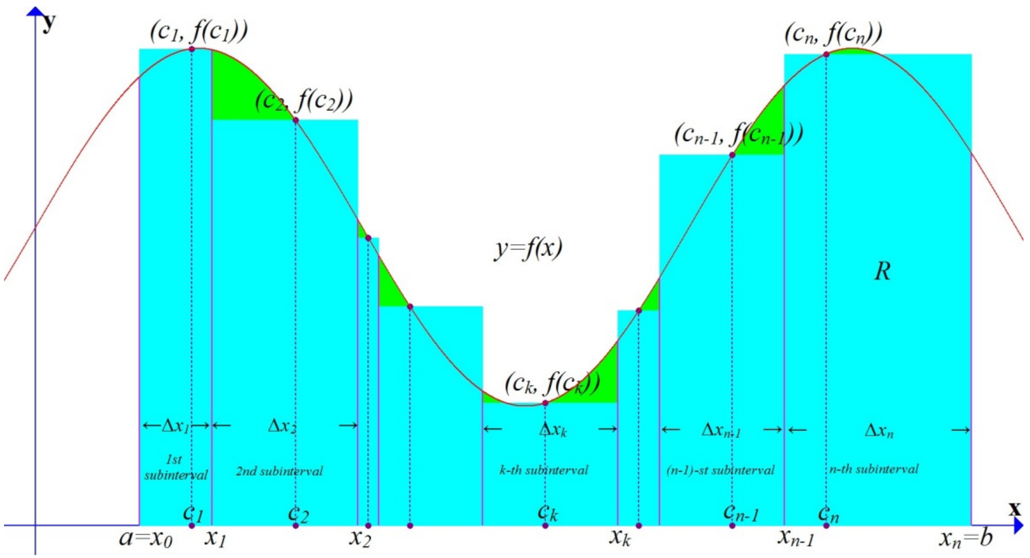
The ***k*-th subinterval of *P*** is $[x_{k-1}, x_k]$ and its width is $\Delta x_k = x_k - x_{k-1}$. The largest of these lengths is called the **norm of *P***, written $\|P\|$.



Riemann Sums

Now, choose arbitrary points c_k in the k-th subinterval for each $k = 1, 2, 3, \cdots, n-1, n$ to get the **Riemann sum**

$$S_P = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + f(c_3)\Delta x_3 + \cdots + f(c_n)\Delta x_n = \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k$$



Example 1

a) For $f(x) = x$, find a formula for the Riemann sum obtained by dividing [1,2] into n equal subintervals using the left endpoint rule.

b) Then take a limit of these sums as $n \to \infty$ to calculate the area under the curve over [1,2].

Example 1 (continued)
$f(x) = x$ over [1,2]

Solution (a): Find the formula for the Riemann sum.

$$f(x) = x$$

$$a = 1 \text{ and } b = 2$$

For all k:

$$\Delta x_k = \Delta x = \frac{\text{width of the interval}}{\text{number of subintervals}}$$

$$= \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$$

Example 1 (continued)
$f(x) = x$ over [1,2]

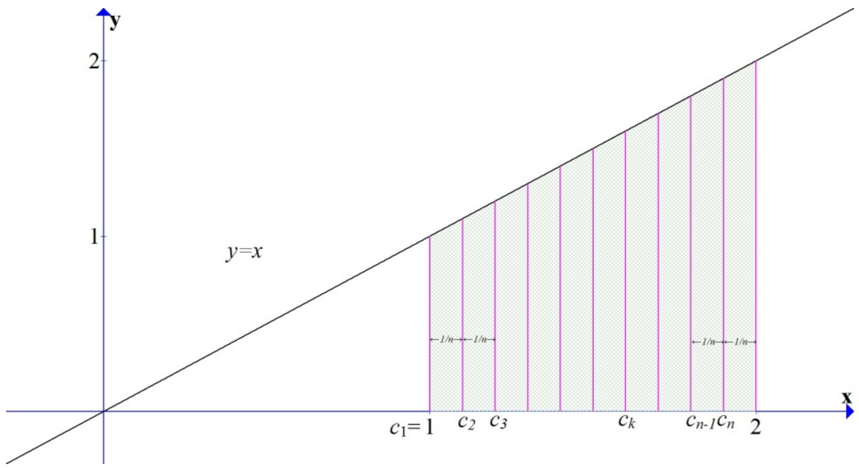
$$c_1 = 1$$

$$c_2 = 1 + \frac{1}{n}$$

$$c_3 = 1 + \frac{2}{n}$$

$$\vdots$$

$$c_k = 1 + \frac{k-1}{n}$$



Example 1 (continued) $f(x) = x$ over [1,2]

$$S_P = \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k$$

$$S_P = \sum_{k=1}^{n} f\left(1 + \frac{k-1}{n}\right) \cdot \frac{1}{n}$$

$$S_P = \sum_{k=1}^{n} \left(1 + \frac{k-1}{n}\right) \cdot \frac{1}{n}$$

Example 1 (continued) $f(x) = x$ over [1,2]

$$S_P = \sum_{k=1}^{n} \left(\frac{1}{n} + \frac{k-1}{n^2} \right)$$

$$S_P = \sum_{k=1}^{n} \frac{1}{n} + \sum_{k=1}^{n} \frac{k-1}{n^2}$$

$$S_P = n \cdot \frac{1}{n} + \frac{1}{n^2} \sum_{k=1}^{n} (k-1)$$

Example 1 (continued)
$f(x) = x$ over [1,2]

$$S_P = 1 + \frac{1}{n^2}\left[\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1\right]$$

$$S_P = 1 + \frac{1}{n^2}\left[\frac{n(n+1)}{2} - n \cdot 1\right]$$

$$S_P = 1 + \frac{n+1}{2n} - \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{2n} - \frac{2}{2n} = \frac{3}{2} - \frac{1}{2n}$$

<u>Answer (a)</u>: $S_P = \frac{3}{2} - \frac{1}{2n}$

Example 1 (continued)
$f(x) = x$ over [1,2]

Solution (b): Take a limit of S_P to calculate the area under the curve.

$$A = \lim_{n\to\infty} S_P = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k) \cdot \Delta x$$

$$A = \lim_{n\to\infty} \left(\frac{3}{2} - \frac{1}{2n}\right) = \frac{3}{2}$$

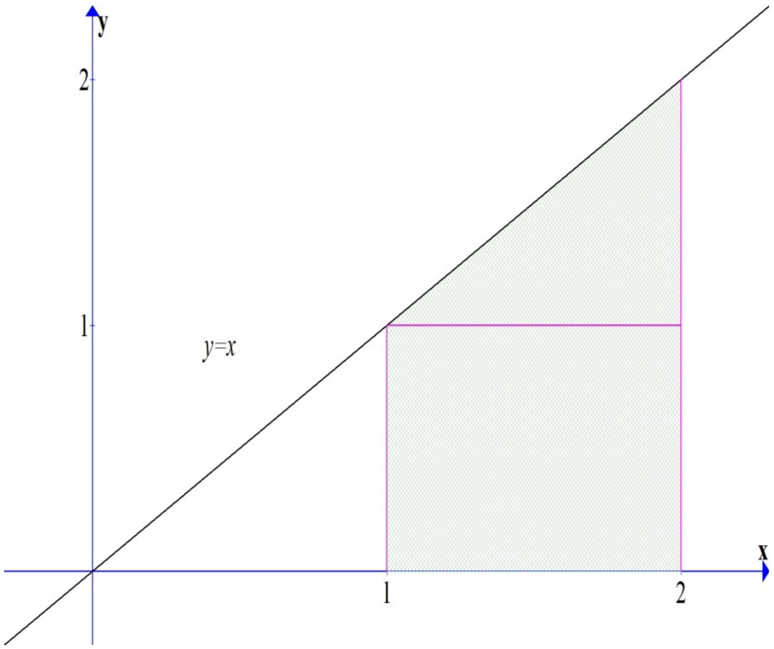
Answer (b):

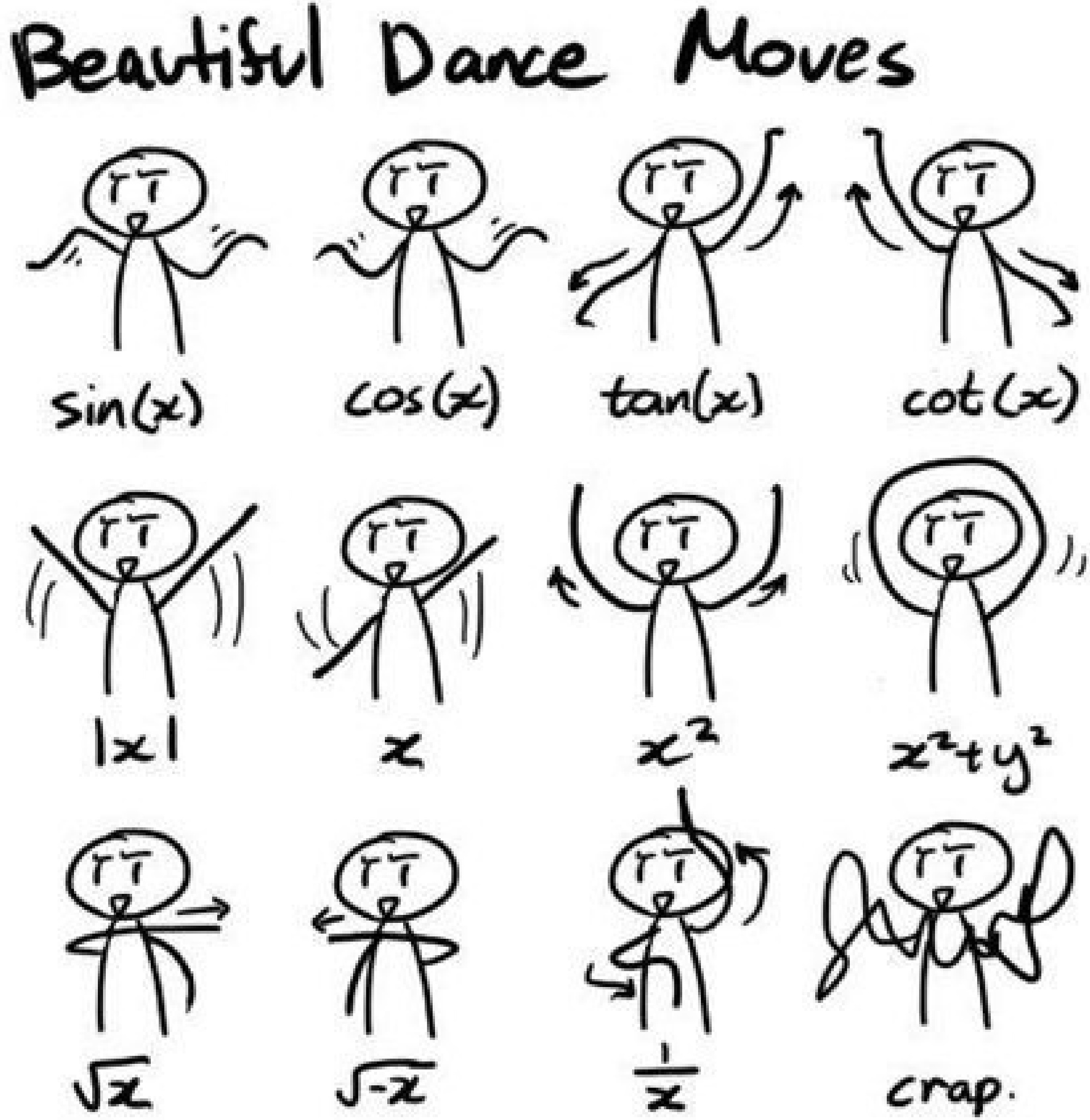
The area under the curve is $A = \frac{3}{2}$.

Example 1 (continued) $f(x) = x$ over [1,2]

This makes since the area under the curve is just a square with sides of length 1 topped with an equilateral right triangle with sides of length 1.

$A = \text{area of square} + \text{area of triangle} = 1 \cdot 1 + \frac{1}{2} \cdot 1 \cdot 1 = \frac{3}{2}.$




Beautiful Dance Moves
sin(x)
cos(x)
tan(x)
cot(x)
|x|
x
x²
x²+y²
√x
√-x
1/x
crap.