

# The Definite Integral

## Part 2

# Recall: Definite Integral

If the function  $f(x)$  is defined on  $[a, b]$ , then the **definite integral of  $f$  from  $a$  to  $b$**  is

$$\int_a^b f(x) dx = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(c_k) \cdot \Delta x_k$$

provided the limit exists.

# Theorems

1. If  $f(x)$  is continuous on  $[a, b]$ , then  $f(x)$  is integrable on  $[a, b]$ .
2. If  $f(x)$  has only finitely many points of discontinuity on  $[a, b]$  and if there is a positive number  $M$  such that  $-M \leq f(x) \leq M$  for all  $x$  in  $[a, b]$  (that is,  $f(x)$  is **bounded** on  $[a, b]$ ), then  $f(x)$  is integrable on  $[a, b]$ .
3. If  $f(x)$  is continuous and assumes both positive and negative values on  $[a, b]$ , then

$$\int_a^b f(x) dx = [\text{area above } x\text{-axis}] - [\text{area below } x\text{-axis}]$$

# Basic Properties of Integrals

a) Zero Width Interval

$$\int_a^a f(x) dx = 0$$

b) Constant Multiple

$$\int_a^b k \cdot f(x) dx = k \int_a^b f(x) dx$$

c) Sum/Difference

$$\int_a^b (f(x) \pm g(x)) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$$

d) Additivity

$$\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$$

# Basic Properties of Integrals (continued)

e) Max-Min Inequality: If  $f(x)$  has maximum value  $\max f$  and minimum value  $\min f$  on  $[a, b]$ , then

$$\min f \cdot (b - a) \leq \int_a^b f(x) dx \leq \max f \cdot (b - a)$$

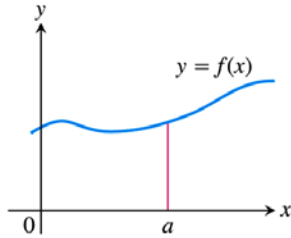
f) Domination

–  $f(x) \geq g(x)$  on  $[a, b] \implies \int_a^b f(x) dx \geq \int_a^b g(x) dx$

–  $f(x) \geq 0$  on  $[a, b] \implies \int_a^b f(x) dx \geq 0$

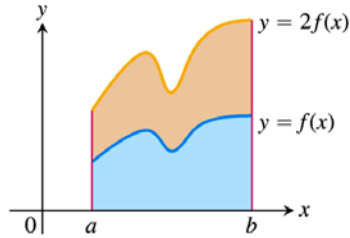
g) Order

$$\int_b^a f(x) dx = - \int_a^b f(x) dx$$



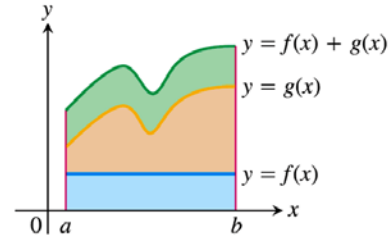
(a) Zero Width Interval:

$$\int_a^a f(x) dx = 0$$



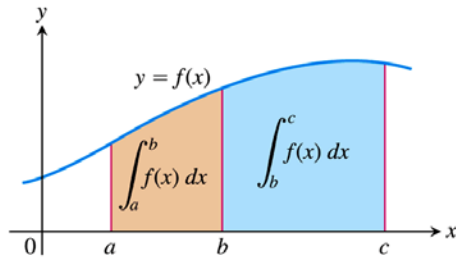
(b) Constant Multiple: ( $k = 2$ )

$$\int_a^b kf(x) dx = k \int_a^b f(x) dx$$



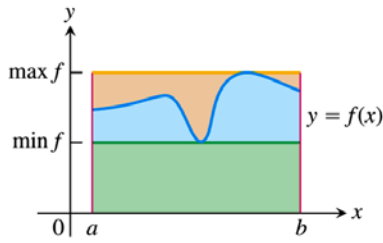
(c) Sum: (areas add)

$$\int_a^b (f(x) + g(x)) dx = \int_a^b f(x) dx + \int_a^b g(x) dx$$



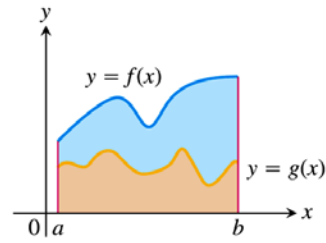
(d) Additivity for definite integrals:

$$\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$$



(e) Max-Min Inequality:

$$\begin{aligned} \min f \cdot (b - a) &\leq \int_a^b f(x) dx \\ &\leq \max f \cdot (b - a) \end{aligned}$$



(f) Domination:

$$\begin{aligned} f(x) &\geq g(x) \text{ on } [a, b] \\ \Rightarrow \int_a^b f(x) dx &\geq \int_a^b g(x) dx \end{aligned}$$

# Basic Definite Integrals

- $\int_a^b x \, dx = \frac{b^2}{2} - \frac{a^2}{2}, \quad a < b$
- $\int_a^b c \, dx = c(b - a), \quad c \text{ any constant}$
- $\int_a^b x^2 \, dx = \frac{b^3}{3} - \frac{a^3}{3}, \quad a < b$

# Example 1

Evaluate:

$$a) \int_7^4 3 dx$$

$$b) \int_0^{15} 7x dx$$

$$c) \int_4^1 (3x + 1) dx$$

$$d) \int_{1/2}^2 5x^2 dx$$



# Example 1 (continued)

Solution (a):

$$\int_7^4 3 dx$$

$$= - \int_4^7 3 dx$$

$$= -[3(7 - 4)] = -9$$

# Example 1 (continued)

Solution (b):

$$\int_0^{15} 7x \, dx$$

$$= 7 \int_0^{15} x \, dx$$

$$= 7 \left( \frac{15^2}{2} - \frac{0^2}{2} \right) = \frac{1575}{2}$$

# Example 1 (continued)

Solution (c):

$$\int_4^1 (3x + 1) dx$$

$$= - \int_1^4 (3x + 1) dx$$

$$= - \left[ \int_1^4 3x dx + \int_1^4 1 dx \right]$$

$$= - \left[ 3 \int_1^4 x dx + \int_1^4 1 dx \right]$$

$$= - \left[ 3 \left( \frac{4^2}{2} - \frac{1^2}{2} \right) + 1(4 - 1) \right] = -\frac{51}{2}$$

# Example 1 (continued)

Solution (d):

$$\int_{1/2}^2 5x^2 dx$$

$$= 5 \int_{1/2}^2 x^2 dx$$

$$= 5 \left( \frac{2^3}{3} - \frac{(1/2)^3}{3} \right) = \frac{105}{8}$$

# Average Value of a Continuous Function

If  $f(x)$  is integrable on  $[a, b]$ , then its **average value on  $[a, b]$** , also called its **mean**, is

$$\text{av}(f) = \frac{1}{b - a} \int_a^b f(x) dx$$

## Example 2

Graph  $f(x) = -x^2$  and find its average value over  $[-1,4]$ .

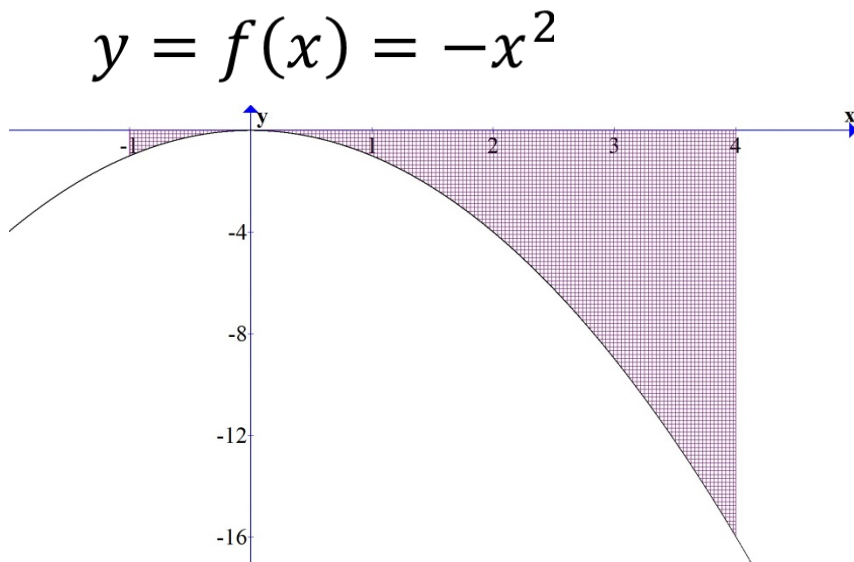
Solution:

$$\text{av}(f) = \frac{1}{b-a} \int_a^b f(x) dx$$

We have:

$$f(x) = -x^2$$
$$[a, b] = [-1, 4]$$

# Example 2 (continued)



$$\begin{aligned} \text{av}(f) &= \frac{1}{b-a} \int_a^b f(x) dx \\ &= \frac{1}{4 - (-1)} \int_{-1}^4 (-x^2) dx \\ &= -\frac{1}{5} \int_{-1}^4 x^2 dx \\ &= -\frac{1}{5} \left( \frac{4^3}{3} - \frac{(-1)^3}{3} \right) \\ &= -\frac{13}{3} \end{aligned}$$

# Eureka!



The exclamation 'Eureka!' is famously attributed to the ancient Greek scholar **Archimedes**. He reportedly proclaimed "Eureka!" when he stepped into a bath and noticed that the water level rose—he suddenly understood that the volume of water displaced must be equal to the volume of the part of his body he had submerged. He then realized that the volume of irregular objects could be measured with precision, a previously intractable problem. He is said to have been so eager to share his discovery that he leapt out of his bathtub and ran through the streets of Syracuse naked.