The Fundamental Theorem of Calculus

Part 1

The Mean Value Theorem for Integrals

If f is continuous on $[a, b]$, then there is at least one number c in $[a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b - a)$$

The Mean Value Theorem for Integrals: Rough Proof

By the Extreme Value Theorem, f assumes a maximum value M and a minimum value m on $[a, b]$. Therefore

$$m(b-a) = \int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx = M(b-a)$$

or

$$m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M$$

It follows from the Intermediate Value Theorem that there exists a $c \in [a, b]$ such that

$$\frac{1}{b-a}\int_a^b f(x)\,dx = f(c)\big(= \mathrm{av}(f)\big)$$

giving us our result.

The Fundamental Theorem of Calculus

(a) Let f be continuous on an open interval I, and let $a \in I$. If

$$F(x) = \int_a^x f(t)\,dt$$

Then

$$F'(x) = \frac{d}{dx}[F(x)] = \frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

(b) If f is continuous on $[a, b]$ and if F is an antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

The Fundamental Theorem of Calculus: Rough Proof of (b)

Let $f(x)$ be a function defined on a closed interval $[a, b]$ and

$$P = \{a = x_0, x_1, x_2, x_3, \cdots, x_{n-1}, x_n = b\}$$

be a partition of the interval.

Since F is an antiderivative of f on $[a, b]$,

$$F'(x) = f(x).$$

The Fundamental Theorem of Calculus: Rough Proof of (b) (continued)

We can write:

$$
\begin{aligned}
F(b) - F(a) \\
&= [F(x_1) - F(a)] + [F(x_2) - F(x_1)] \\
&+ [F(x_3) - F(x_2)] + \cdots + [F(b) - F(x_{n-1})]
\end{aligned}
$$

Using the Mean Value Theorem, we can find a $c_k \in (x_{k-1}, x_k)$ such that

$$F'(c_k) = \frac{F(x_k) - F(x_{k-1})}{x_k - x_{k-1}}$$

Or,

$$F'(c_k) \cdot (x_k - x_{k-1}) = F(x_k) - F(x_{k-1}).$$

The Fundamental Theorem of Calculus: Rough Proof of (b) (continued)

So the equation above can be rewritten as:

$$
\begin{aligned}
F(b) - F(a) \\
&= F'(c_1)(x_1 - a) + F'(c_2)(x_2 - x_1) \\
&+ F'(c_3)(x_3 - x_2) + \cdots + F'(c_n)(b - x_{n-1}) \\
&= f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + f(c_3)\Delta x_3 + \cdots \\
&+ f(c_n)\Delta x_n \\
&= \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k
\end{aligned}
$$

The Fundamental Theorem of Calculus: Rough Proof of (b) (continued)

Since

$$\lim_{\|P\|\to 0}\big(F(b)-F(a)\big)=F(b)-F(a)$$

and

$$\lim_{\|P\|\to 0}\sum_{k=1}^{n} f(c_k)\cdot\Delta x_k=\int_a^b f(x)\,dx,$$

we get our result.

Notation

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big]_a^b$$

Note: $\int_a^b f(x)\,dx = \int f(x)\,dx\Big]_a^b$, so this shows how definite and indefinite integrals are related.

Example 1

Evaluate

$$\int_1^2 x\,dx$$

<u>Solution</u>:

$$\int_1^2 x\,dx$$

$$= \frac{1}{2}x^2\Big]_1^2$$

$$= \frac{1}{2}\cdot 2^2 - \frac{1}{2}\cdot 1^2$$

$$= \frac{3}{2}$$

Example 2

Find the area under the curve $y = \cos(x)$ over the interval $\left[0, \frac{\pi}{2}\right]$.

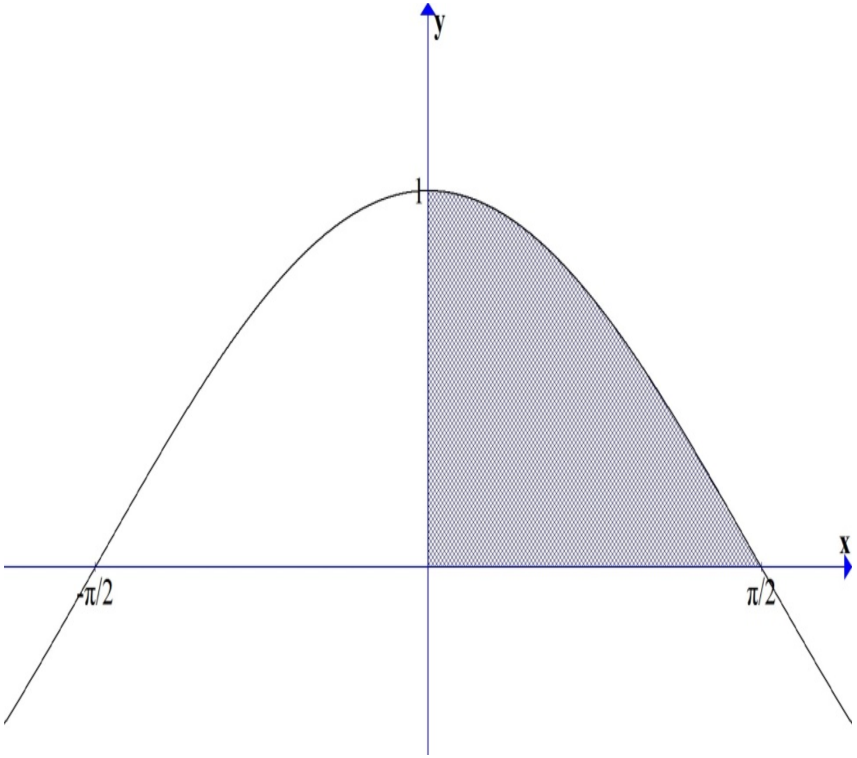
Solution:

$$A = \int_0^{\pi/2} \cos(x)\, dx$$

$$= \sin(x)\Big]_0^{\pi/2}$$

$$= \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1$$

$y = \cos(x)$



Example 3

Evaluate

$$\int_0^2 \frac{9}{2} x^2 \sqrt{x^3 + 1}\, dx$$

<u>Solution</u>:

Notice:

$$\frac{d}{dx}\left((x^3 + 1)^{3/2}\right) = \frac{9}{2} x^2 \sqrt{x^3 + 1}$$

Example 3 (continued)

So by the Fundamental Theorem of Calculus:

$$\int_0^2 \frac{9}{2} x^2 \sqrt{x^3 + 1}\, dx$$

$$= (x^3 + 1)^{3/2} \Big]_0^2$$

$$= (2^3 + 1)^{3/2} - (0^3 + 1)^{3/2} = 26$$

"Guess you should have let me check your math when you ordered the mulch."

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