The Fundamental Theorem of Calculus

Part 2

Recall: The Fundamental Theorem of Calculus

(a) Let f be continuous on an open interval I, and let $a \in I$. If

$$F(x) = \int_a^x f(t)\,dt$$

Then

$$F'(x) = \frac{d}{dx}[F(x)] = \frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

(b) If f is continuous on $[a, b]$ and if F is an antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Example 1

Find

$$\frac{d}{dx}\left[\int_0^{\sqrt{x}} \cos(t)\, dt\right]$$

Example 1 (continued)

Solution:

$$\frac{d}{dx}\left[\int_0^{\sqrt{x}} \cos(t)\, dt\right] = \frac{d}{dx}\left[f\big(g(x)\big)\right]$$

where

$$f(x) = \int_0^{x} \cos(t)\, dt$$

and

$$g(x) = \sqrt{x}$$

Example 1 (continued)

By the Chain Rule:

$$\frac{d}{dx}\left[\int_0^{\sqrt{x}} \cos(t)\,dt\right] = \frac{d}{dx}\left[f\big(g(x)\big)\right] = f'\big(g(x)\big)\cdot g'(x)$$

Now

$$f'(x) = \frac{d}{dx}\left[\int_0^{x} \cos(t)\,dt\right] = \cos(x)$$

$$f'\big(g(x)\big) = \cos\big(g(x)\big) = \cos\sqrt{x}$$

and

$$g'(x) = \frac{d}{dx}(\sqrt{x}) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}.$$

Example 1 (continued)

Therefore

$$\frac{d}{dx}\left[\int_0^{\sqrt{x}} \cos(t)\,dt\right]$$

$$= \cos(\sqrt{x}) \cdot \left(\frac{1}{2\sqrt{x}}\right)$$

$$= \frac{\cos\sqrt{x}}{2\sqrt{x}}$$

Discovery from Example 1

$$\frac{d}{dx}\left[\int_a^{g(x)} f(t)\,dt\right] = f\big(g(x)\big) \cdot g'(x)$$

Example 2

Find

$$\frac{d}{dx}\left[\int_{\tan(x)}^{0} \frac{1}{1+t^2}\,dt\right]$$

Solution:

$$\frac{d}{dx}\left[\int_{\tan(x)}^{0} \frac{1}{1+t^2}\,dt\right]$$

$$= \frac{d}{dx}\left[-\int_{0}^{\tan(x)} \frac{1}{1+t^2}\,dt\right]$$

Example 2 (continued)

$$\frac{d}{dx}\left[-\int_0^{\tan(x)} \frac{1}{1+t^2}\,dt\right]$$

$$= -\frac{1}{1+(\tan(x))^2}\cdot\frac{d}{dx}(\tan(x))$$

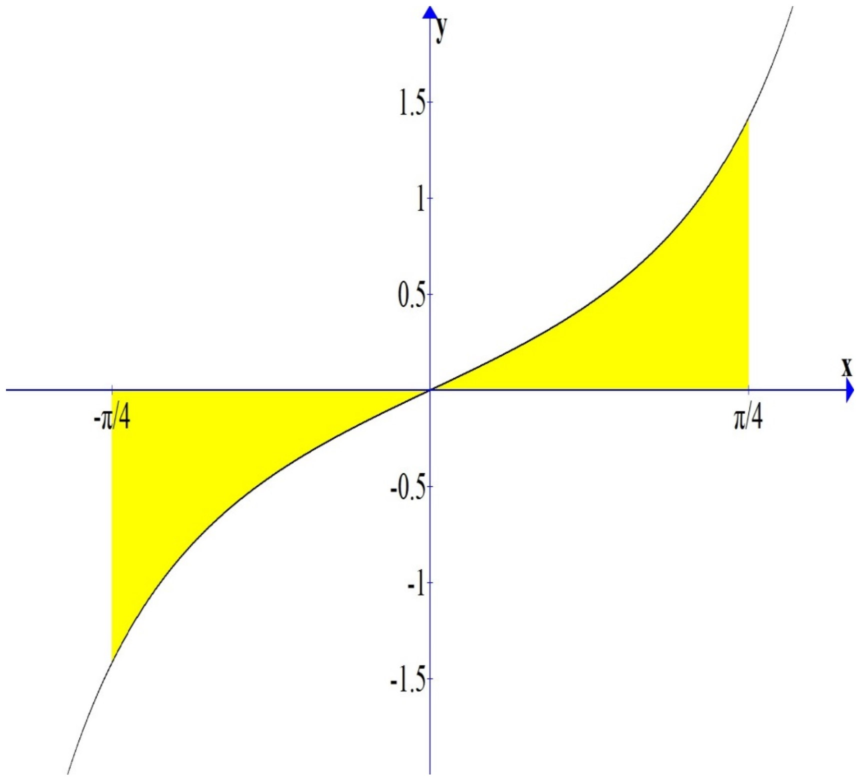
$$= -\frac{\sec^2(x)}{1+\tan^2(x)}$$

$$= -\frac{\sec^2(x)}{\sec^2(x)} = -1$$

Example 3

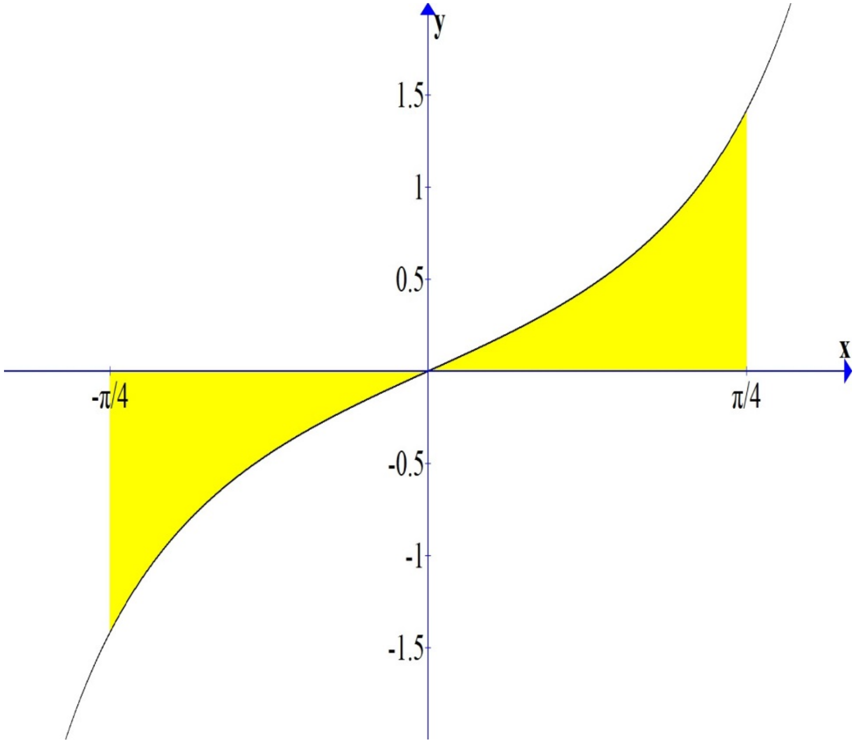
Find the total area between the region $y = \sec(x)\tan(x), -\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$ and the x-axis.

$y = \sec(x)\tan(x)$



Example 3 (continued)

$y = \sec(x)\tan(x)$



Solution:

In order to find the total area, we will need to calculate the area below the x-axis and the area above the x-axis.

Example 3 (continued)

$A_1 =$ area under the $x-$axis

$$= -\int_{-\pi/4}^{0} \sec(x)\tan(x)\,dx$$

$$= -\sec(x)\Big]_{-\pi/4}^{0}$$

$$= -\sec(0) - \left(-\sec\left(-\frac{\pi}{4}\right)\right) = -1 + \sqrt{2}$$

Example 3 (continued)

$$A_2 = \text{area above the } x\text{−axis}$$

$$= \int_0^{\pi/4} \sec(x)\tan(x)\,dx$$

$$= \sec(x)\Big]_0^{\pi/4}$$

$$= \sec\left(\frac{\pi}{4}\right) - \sec(0) = \sqrt{2} - 1$$

Example 6 (continued)

Answer:

The total area is:

$$A = A_1 + A_2$$
$$= \left(-1 + \sqrt{2}\right) + \left(\sqrt{2} - 1\right)$$
$$= 2\sqrt{2} - 2$$

Displacement and Distance Traveled

Suppose $v(t)$ is the velocity of an object moving along a straight line and that $s(t)$ is the position.

$$s = \int v\,dt$$

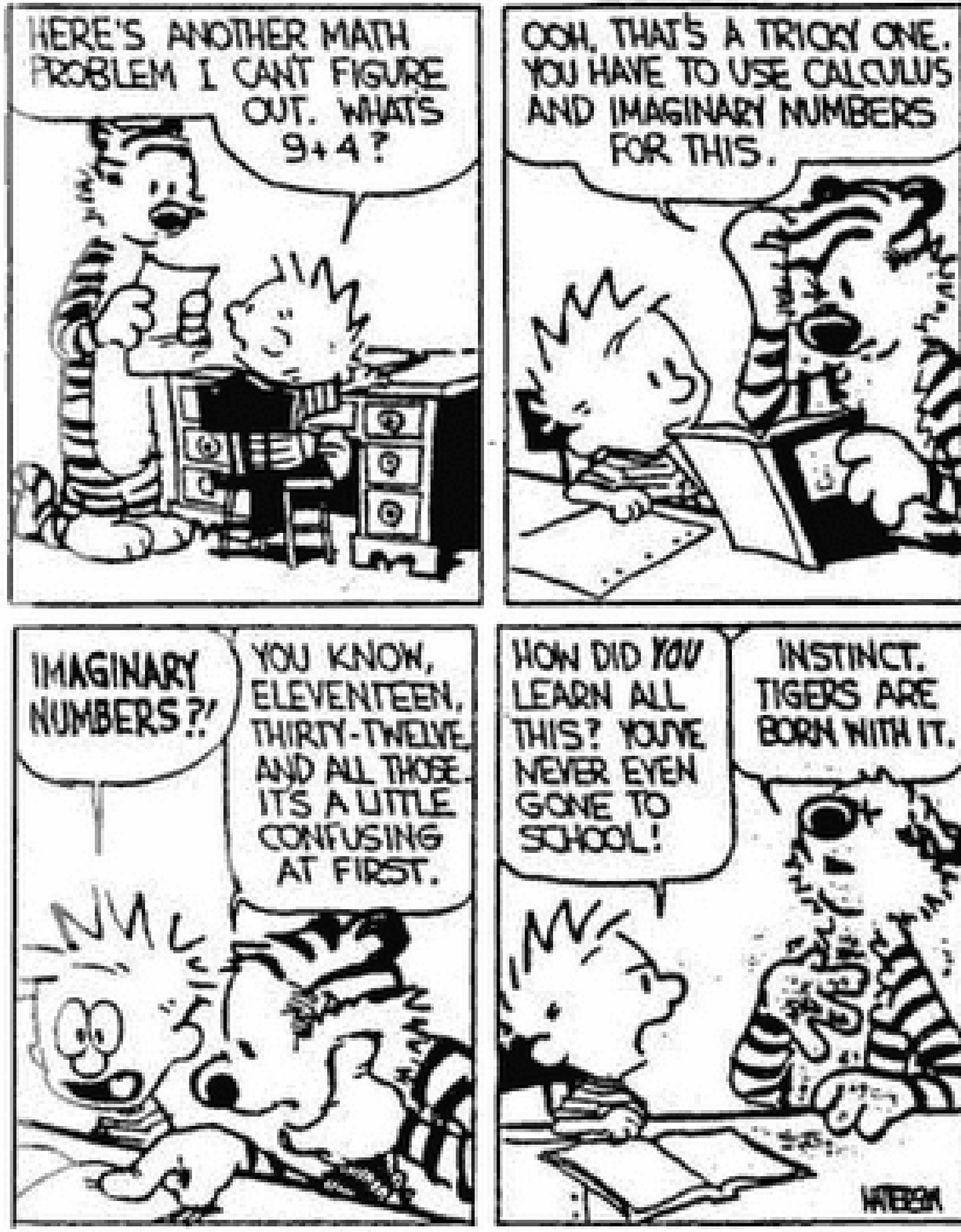
It follows from the First Fundamental Theorem of Calculus that the displacement of that object over the time interval $[a, b]$ is

$$s(b) - s(a) = \int_a^b v(t)\,dt$$

Displacement and Distance Traveled

To find the total distance traveled, which is the distance traveled in the positive direction plus the distance traveled in the negative direction, we have:

$$\text{total distance traveled during time interval } [a, b] = \int_a^b |v(t)|\, dt$$


HERE'S ANOTHER MATH PROBLEM I CAN'T FIGURE OUT. WHAT'S 9+4?
OOH, THAT'S A TRICKY ONE. YOU HAVE TO USE CALCULUS AND IMAGINARY NUMBERS FOR THIS.
IMAGINARY NUMBERS?!
YOU KNOW, ELEVENTEEN, THIRTY-TWELVE AND ALL THOSE. IT'S A LITTLE CONFUSING AT FIRST.
HOW DID YOU LEARN ALL THIS? YOU'VE NEVER EVEN GONE TO SCHOOL!
INSTINCT. TIGERS ARE BORN WITH IT.
WATERSON