Indefinite Integrals and the Substitution Method

Part 1

Substitution Method

$$\int \left(f(u) \frac{du}{dx} \right) dx = \int f(u)\, du$$

Reason:

Suppose $\frac{d}{du}[F(u)] = f(u)$. Then

$$\int f(u)\, du = F(u) + C \text{ (Equation 1)}$$

By the Chain Rule

$$\frac{d}{dx}[F(u)] = \frac{d}{du}[F(u)] \cdot \frac{du}{dx} = f(u) \cdot \frac{du}{dx}$$

Giving us

$$\int \left(f(u) \frac{du}{dx} \right) dx = \int \left(\frac{d}{dx}[F(u)] \right) dx = F(u) + C \text{ (Equation 2)}$$

Since Equation 1 and Equation 2 are both equal to $F(u) + C$, we get our result.

Substitution Method

Suppose we are interested in evaluating

$$\int h(x)\,dx$$

It follows that if we can express this integral in the form

$$\int h(x)\,dx = \int f\big(g(x)\big)g'(x)\,dx$$

then the substitution $u = g(x)$ and $du/dx = g'(x)$ will yield

$$\int h(x)\,dx = \int \left[f(u)\frac{du}{dx}\right]dx = \int f(u)\,du$$

Substitution Method

- Step 1: Make a choice for $u = g(x)$.
- Step 2: Compute $\frac{du}{dx} = g'(x)$.
- Step 3: Make the substitution $u = g(x)$ and $du = g'(x)dx$.
- Now the entire integral must be in terms of u. If there are any x's, try a different choice of u.
- Step 4: Evaluate the integral.
- Step 5: Replace u by $g(x)$, so the final answer is in terms of x.

Example 1

Evaluate

$$\int (x^2 + 1)^{50} 2x \, dx$$

<u>Solution</u>:

Step 1: Make a choice for $u = g(x)$.

$$u = x^2 + 1$$

Example 1 (continued)

Step 2: Compute $\frac{du}{dx} = g'(x)$.

$$\frac{du}{dx} = \frac{d}{dx}(x^2 + 1) = 2x$$

Step 3: Make the substitution $u = g(x)$ and $du = g'(x)dx$.

Now the entire integral must be in terms of u. If there are any x's, try a different choice of u.

Example 1 (continued)

$$u = x^2 + 1$$
$$du = 2x \cdot dx$$

$$\int \underbrace{(x^2 + 1)}_{u}{}^{50} \underbrace{2x\,dx}_{du}$$

$$= \int u^{50}\,du$$

Example 1 (continued)

Step 4: Evaluate the integral.

$$\int (x^2+1)^{50} 2x\,dx = \int u^{50}\,du = \frac{1}{51}u^{51} + C$$

Step 5: Replace u by $g(x)$, so the final answer is in terms of x.

$$\int (x^2+1)^{50} 2x\,dx = \frac{1}{51}u^{51} + C = \frac{1}{51}(x^2+1)^{51} + C$$

Example 2

Evaluate

$$\int \sin^2(x)\cos(x)\,dx$$

<u>Solution</u>:

$$u = \sin(x)$$
$$du = \cos(x)\,dx$$

$$\int \sin^2(x)\cos(x)\,dx$$

$$= \int u^2\,du = \frac{1}{3}u^3 + C$$

$$= \frac{1}{3}(\sin(x))^3 + C = \frac{1}{3}\sin^3(x) + C$$

Example 3

Evaluate

$$\int \frac{\cos\sqrt{x}}{2\sqrt{x}}\,dx$$

<u>Solution</u>:

$$u = \sqrt{x} = x^{1/2}$$
$$du = \frac{1}{2}x^{-1/2}dx = \frac{1}{2\sqrt{x}}dx$$

$$\int \frac{\cos\sqrt{x}}{2\sqrt{x}}\,dx = \int \cos\sqrt{x}\cdot\frac{1}{2\sqrt{x}}\,dx$$

$$= \int \cos(u)\cdot du = \sin(u) + C$$

$$= \sin\sqrt{x} + C$$

Example 4

Evaluate

$$\int 3x^2\sqrt{x^3+1}\,dx$$

<u>Solution</u>:

$$u = x^3 + 1$$
$$du = 3x^2 dx$$

$$\int 3x^2\sqrt{x^3+1}\,dx = \int \sqrt{x^3+1}\cdot 3x^2\,dx$$

$$= \int \sqrt{u}\,du = \int u^{1/2}\,du = \frac{2}{3}u^{3/2} + C$$

$$= \frac{2}{3}(x^3+1)^{3/2} + C$$

Example 5

Evaluate

$$\int \cos(5x)\,dx$$

Solution:

$$u = 5x$$
$$du = 5dx \Rightarrow \frac{1}{5}du = dx$$

$$\int \cos(5x)\,dx$$

$$= \int \cos(u) \cdot \frac{1}{5}du = \frac{1}{5}\sin(u) + C$$

$$= \frac{1}{5}\sin(5x) + C$$

Example 6

Evaluate

$$\int \frac{dx}{\left(\frac{1}{3}x - 8\right)^5}$$

<u>Solution</u>:

$$u = \frac{1}{3}x - 8$$
$$du = \frac{1}{3}dx \Rightarrow 3du = dx$$

$$\int \frac{dx}{\left(\frac{1}{3}x - 8\right)^5}$$
$$= \int \frac{3du}{u^5} = \int 3u^{-5}\,du = 3 \cdot \left(-\frac{1}{4}\right)u^{-4} + C$$
$$= -\frac{3}{4}\left(\frac{1}{3}x - 8\right)^{-4} + C$$

Example 7

Evaluate

$$\int x^4 \cdot \sqrt[3]{3-5x^5}\, dx$$

<u>Solution</u>:

$$u = 3 - 5x^5$$

$$du = -25x^4 dx \Rightarrow -\frac{1}{25} du = x^4 dx$$

$$\int x^4 \cdot \sqrt[3]{3-5x^5}\, dx = \int \sqrt[3]{3-5x^5} \cdot x^4\, dx$$

$$= \int \sqrt[3]{u} \left(-\frac{1}{25} du\right) = \int \left(-\frac{1}{25}\right) u^{1/3} du = -\frac{1}{25} \cdot \frac{3}{4} u^{4/3} + C$$

$$= -\frac{3}{100} (3-5x^5)^{4/3} + C$$

Example 8

Evaluate

$$\int x^2 \cdot \sqrt{x-1}\,dx$$

<u>Solution</u>:

$$\boxed{\begin{aligned} u &= x-1 \\ du &= dx \end{aligned}}$$

$$\int x^2 \cdot \sqrt{x-1}\,dx = \int x^2 \sqrt{u}\,du \text{ --Oh–oh!}$$

Example 8 (continued)

$$\int x^2 \cdot \sqrt{x-1}\, dx$$

$$\boxed{\begin{array}{c} u = x - 1 \Rightarrow u + 1 = x \\ du = dx \end{array}}$$

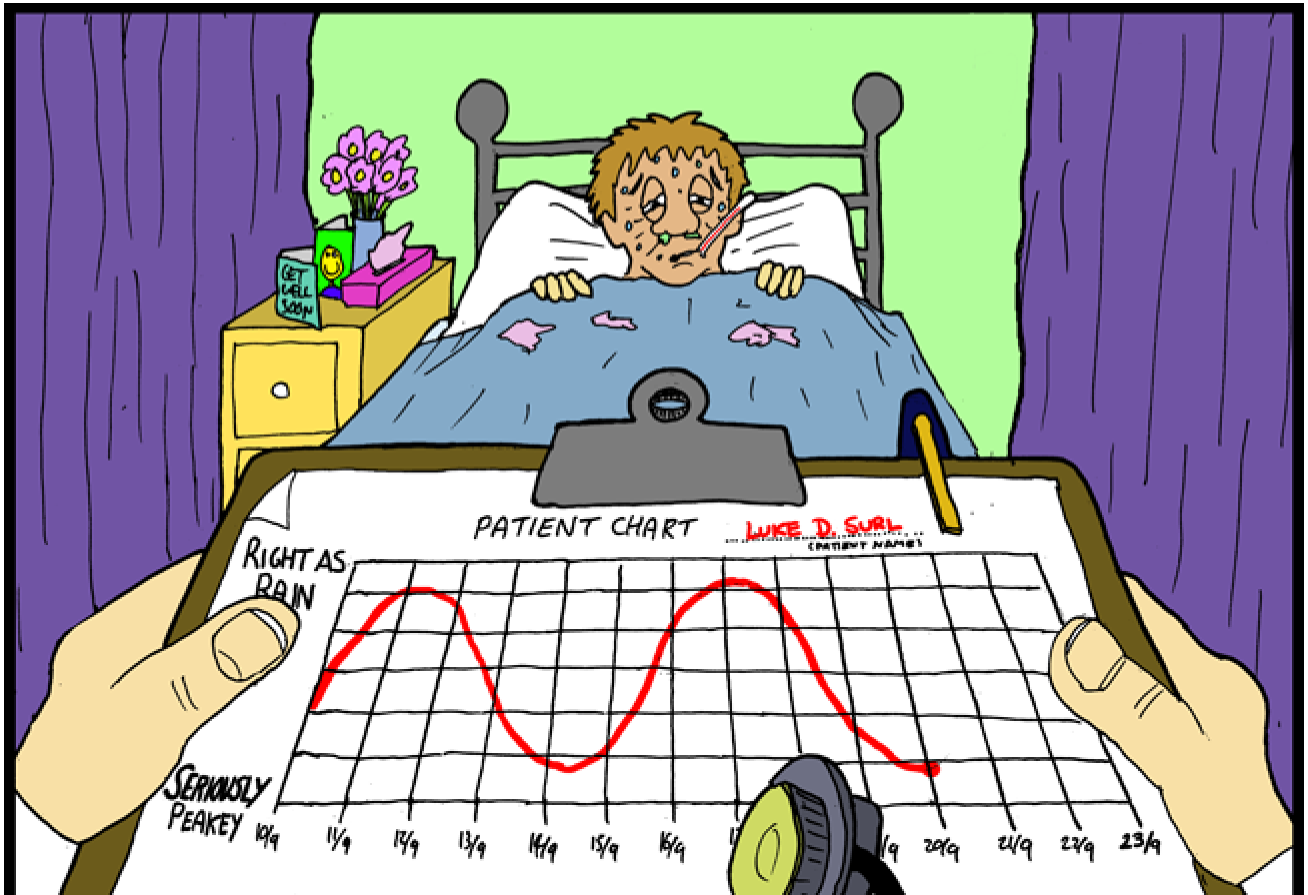
$$\int x^2 \cdot \sqrt{x-1}\, dx$$

$$= \int (u+1)^2 \sqrt{u}\, du$$

$$= \int (u^2 + 2u + 1) u^{1/2}\, du = \int \left(u^{5/2} + 2u^{3/2} + u^{1/2}\right) du$$

$$= \frac{2}{7} u^{7/2} + 2 \cdot \frac{2}{5} u^{5/2} + \frac{2}{3} u^{3/2} + C$$

$$= \frac{2}{7}(x-1)^{7/2} + \frac{4}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C$$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