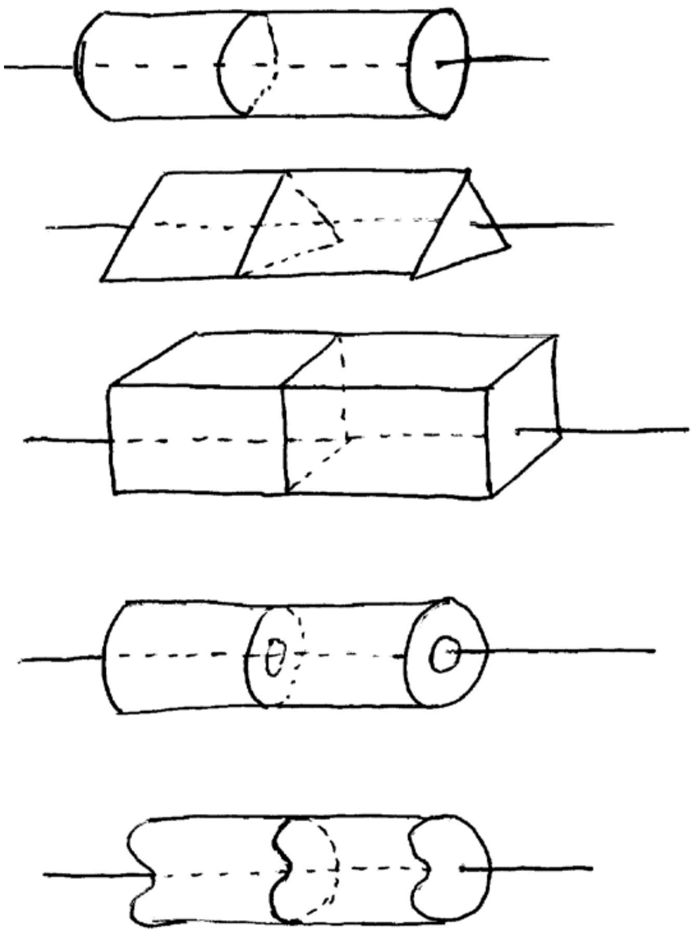
Volumes Using Cross-Sections

Part 1

Volumes of Cylinders and Cross Sections Perpendicular to an Axis

Cylinders

A **right cylinder** is any solid that can generated by moving a plane region along an axis perpendicular to the region.



Volume of a Cylinder

The **volume,** V, of a right cylinder is

the area, A, of the plane region

times

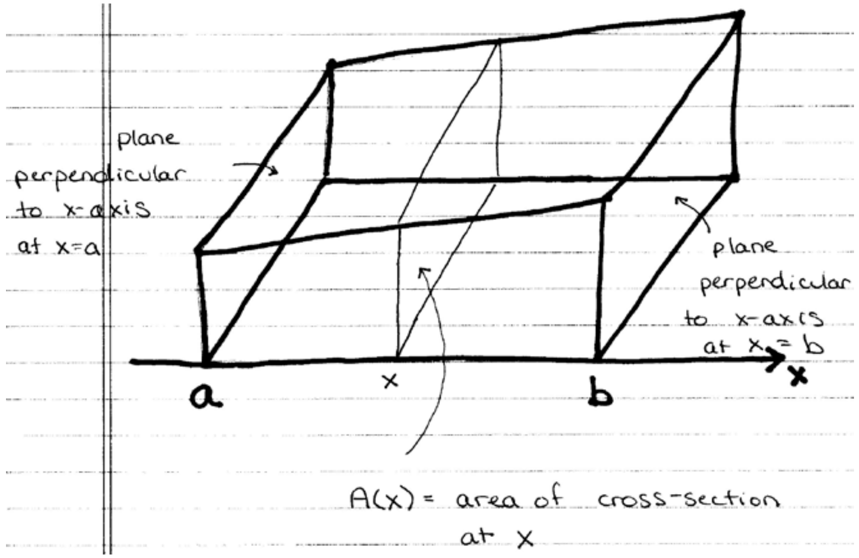
the distance, h, that the plan region has moved along the axis.

$$V = A \cdot h$$

Volumes of Non-cylinders

To find the volume, V, of an object that is not a right cylinder, we use **slicing**.

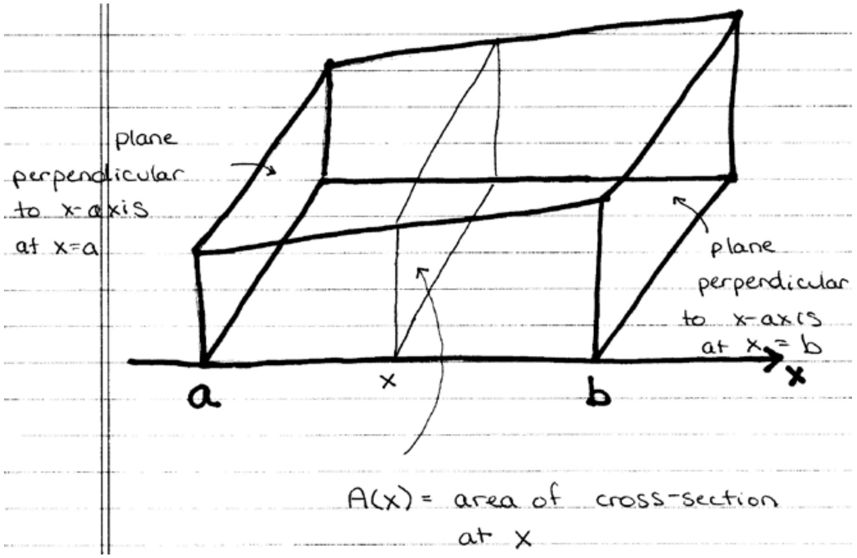
Suppose a solid extends along the x-axis and is bounded on the left and right by planes perpendicular to the x-axis at $x = a$ and $x = b$.



Volumes of Non-cylinders

The cross sections perpendicular to the x-axis can vary from point to point.

$A(x) =$ area of the cross section at x

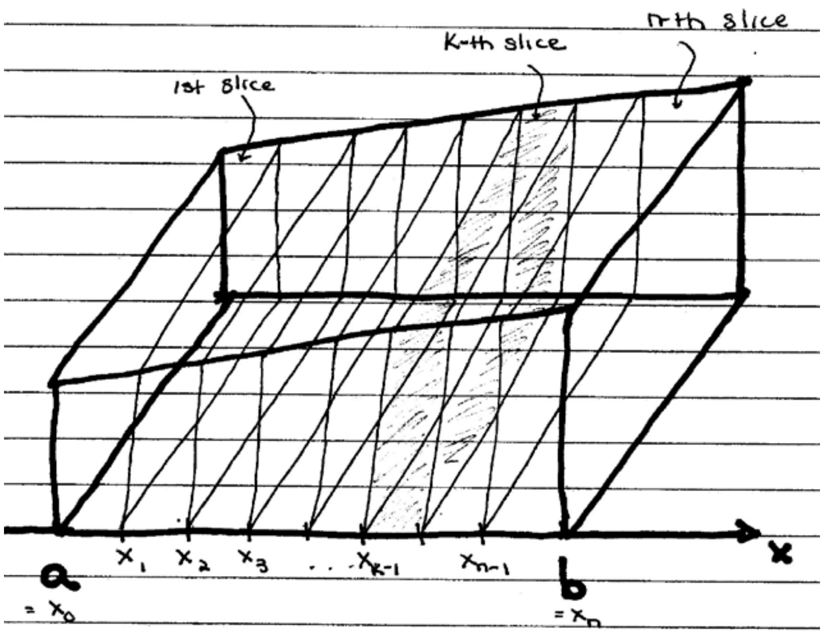


Volumes of Non-cylinders

Let
$P = \{a = x_0, x_1, x_2, x_3, \cdots, x_n = b\}$
be a partition of $[a, b]$.

Pass a plane perpendicular to the x-axis through each of the points in P.

These planes cut the solid into n slices.

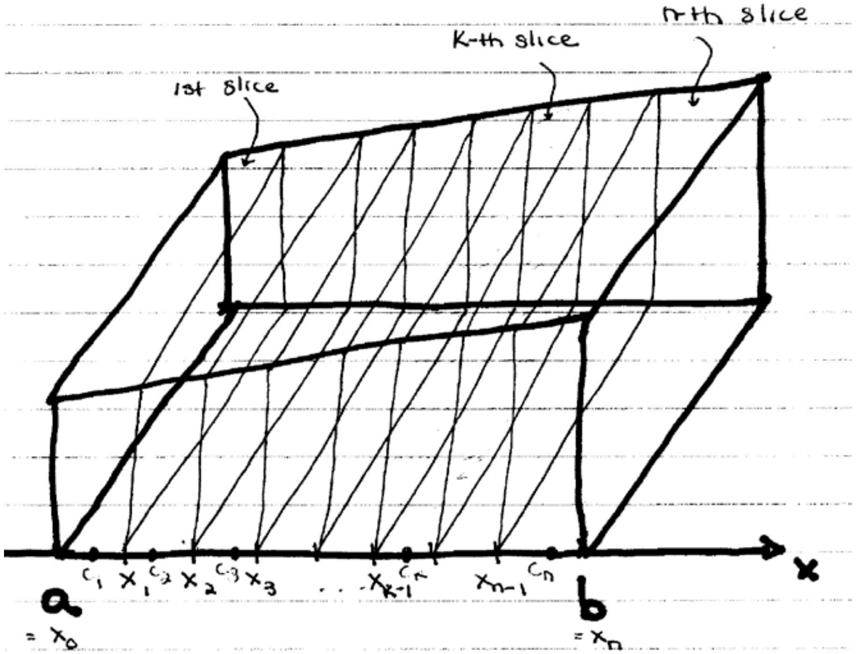


Volumes of Non-cylinders

If the slices are very thin, the cross section is very close to a right cylinder.

To approximate the volume, V_k, of the k-th slice, we choose a c_k in the k-th subinterval of $[a, b]$.

$$V_k \approx A(c_k) \cdot \Delta x_k$$



Volumes of Non-cylinders

$$V = V_1 + V_2 + V_3 + \cdots + V_n$$
$$\approx \sum_{k=1}^{n} A(c_k) \cdot \Delta x_k$$

Therefore,

$$V = \lim_{\|P\| \to 0} \sum_{k=1}^{n} A(c_k) \cdot \Delta x_k = \int_a^b A(x)\, dx$$

Volumes by Cross Sections Perpendicular to the x-Axis

Let S be a solid bounded by two parallel planes perpendicular to the x-axis at $x = a$ and $x = b$.

If, for each x in $[a, b]$, the cross-sectional area of S perpendicular to the x-axis is $A(x)$,

then the volume of the solid is

$$V = \int_a^b A(x)\, dx$$

provided $A(x)$ is integrable.

Volumes by Cross Sections Perpendicular to the y-Axis

Let S be a solid bounded by two parallel planes perpendicular to the y-axis at $y = c$ and $y = d$. If, for each y in $[c, d]$, the cross-sectional area of S perpendicular to the y-axis is $A(y)$, then the volume of the solid is

$$V = \int_c^d A(y)\, dy$$

Provided $A(y)$ is integrable.

Example

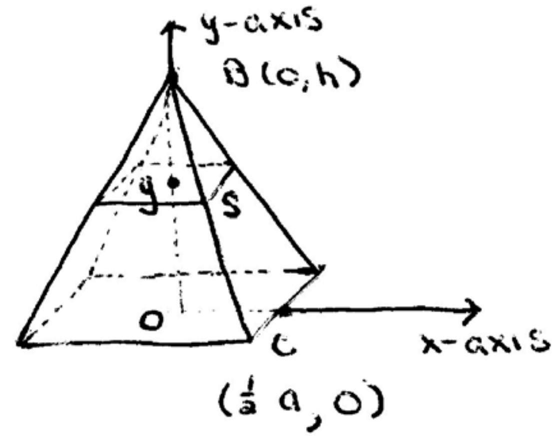
Derive the formula for the volume of a right pyramid whose altitude is h and whose base is a square with sides of length a.

Example (continued)

Solution:

Introduce a rectangular coordinate system so that the y-axis passes through the apex, and the x-axis passes through the base and is parallel to a side of the base.

At any point y in the interval $[0, h]$ on the y-axis the cross section perpendicular to the y-axis is a square.



Example (continued)

$y =$ height of slice

$s =$ length of side of the slice

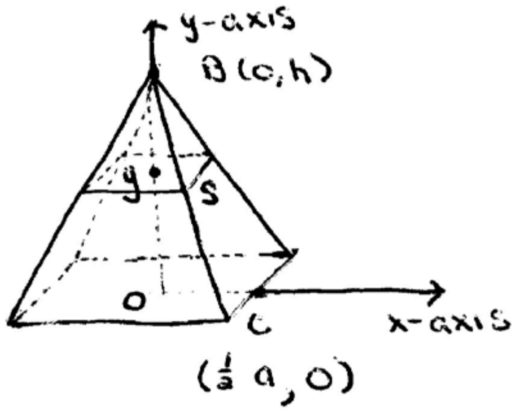
h and a are **constant**

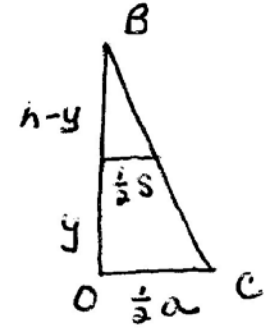
By similar triangles

$$\frac{\frac{1}{2}s}{\frac{1}{2}a} = \frac{h-y}{h}$$

or

$$s = \frac{a}{h}(h-y)$$





Example (continued)

$$A(y) = \text{area of cross section}$$

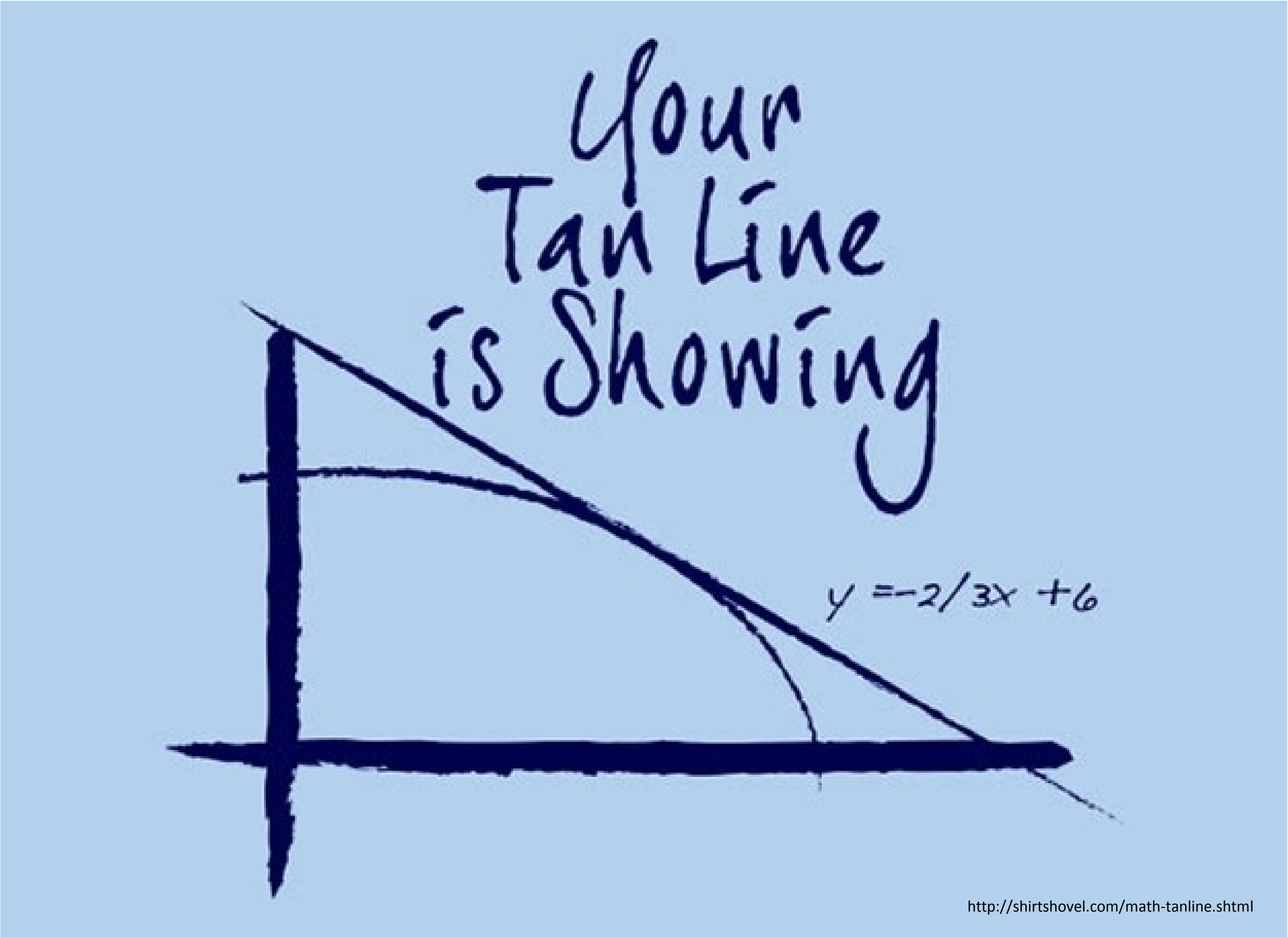
$$A(y) = s^2 = \left[\frac{a}{h}(h-y)\right]^2$$

$$V = \int_0^h A(y)\,dy = \int_0^h \left[\frac{a}{h}(h-y)\right]^2 dy$$

$$= \int_0^h \frac{a^2}{h^2}(h-y)^2\,dy = \frac{a^2}{h^2}\int_0^h (h-y)^2\,dy$$

$$= \frac{a^2}{h^2}\cdot\left(-\frac{1}{3}\right)(h-y)^3\Big|_0^h$$

$$= -\frac{a^2}{3h^2}\left[(h-h)^3-(h-0)^3\right] = \frac{a^2h}{3}$$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