Areas of Surfaces of Revolution

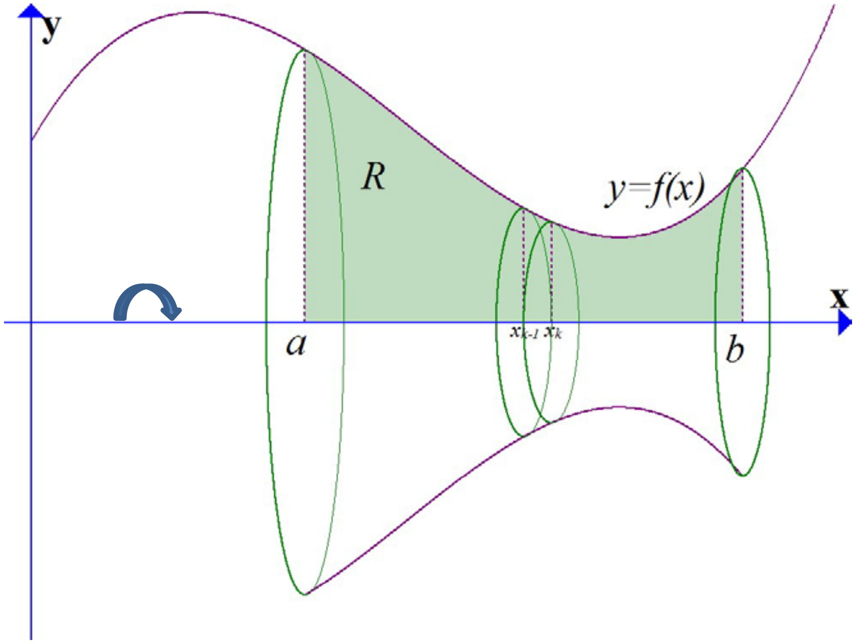
Surface Area

Let f be a smooth, nonnegative function on an interval $[a, b]$.

<u>Problem</u>:

Find the area of the surface generated by revolving the curve $y = f(x)$ about the x-axis.

Surface Area

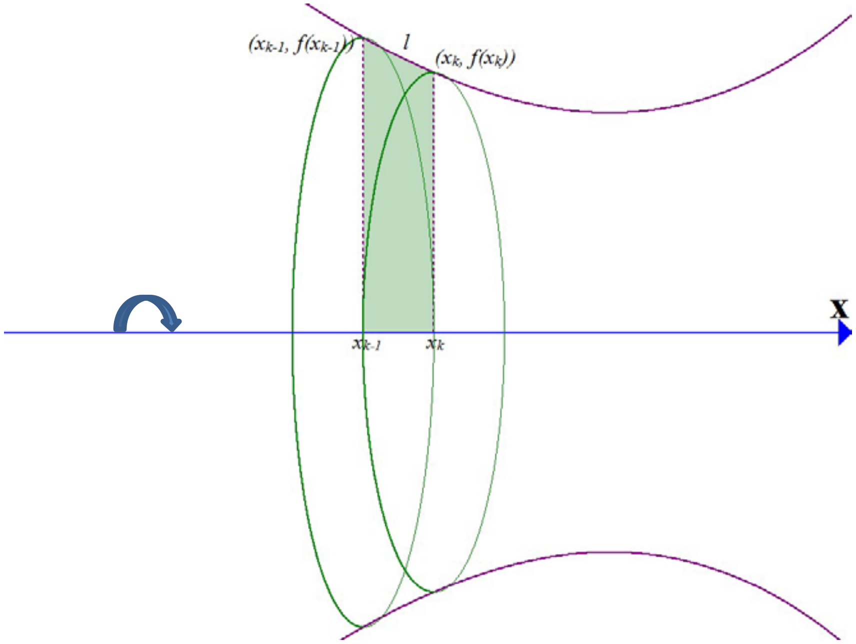


Let f be a nonnegative, smooth function on $[a, b]$,

and
$P =$
$\{a = x_0, x_1, x_2, \cdots, x_{n-1}, x_n = b\}$
be a partition of $[a, b]$.

A slice of the surface generated by revolving the curve about the x-axis is like a **frustum** (the portion of a solid that lies between two parallel planes cutting it) of a cone.

Surface Area



The lateral area of the frustum can be obtained from the formula

$$\pi\big(f(x_{k-1}) + f(x_k)\big) \cdot l$$

where l is the slant height (that is, l is the distance between the points $(x_{k-1}, f(x_{k-1}))$ and $(x_k, f(x_k))$).

Surface Area

$$S_k \approx \pi\big(f(x_{k-1}) + f(x_k)\big) \cdot l$$
$$= \pi\big(f(x_{k-1}) + f(x_k)\big)\sqrt{(\Delta x_k)^2 + \big(f(x_k) - f(x_{k-1})\big)^2}$$

By the Mean-Value Theorem, there is a point c_k between x_{k-1} and x_k such that

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = f'(c_k)$$

or

$$f(x_k) - f(x_{k-1}) = f'(c_k)\Delta x_k$$

This gives us

$$S_k \approx \pi\big(f(x_{k-1}) + f(x_k)\big)\sqrt{(\Delta x_k)^2 + (f'(c_k)\Delta x_k)^2}$$
$$= \pi\big(f(x_{k-1}) + f(x_k)\big)\sqrt{1 + (f'(c_k))^2} \cdot \Delta x_k$$

Surface Area

Now

$$\frac{1}{2}\left(f(x_{k-1}) + f(x_k)\right)$$

is between $f(x_{k-1})$ and $f(x_k)$.

By the Intermediate Value Theorem, we know that there exists a d_k in $[x_{k-1}, x_k]$ such that

$$\frac{1}{2}\left(f(x_{k-1}) + f(x_k)\right) = f(d_k)$$

Surface Area

$$S_k \approx \pi\big(f(x_{k-1}) + f(x_k)\big)\sqrt{1 + (f'(c_k))^2} \cdot \Delta x_k$$
$$= 2\pi f(d_k)\sqrt{1 + (f'(c_k))^2} \cdot \Delta x_k$$

That means that the total surface area, S, is approximately

$$S = \sum_{k=1}^{n} S_k \approx \sum_{k=1}^{n} 2\pi f(d_k)\sqrt{1 + (f'(c_k))^2} \cdot \Delta x_k$$

Surface Area

We expect that

$$S = \lim_{\|P\| \to 0} \sum_{k=1}^{n} 2\pi f(d_k)\sqrt{1 + (f'(c_k))^2} \cdot \Delta x_k$$

If $c_k = d_k$, then this would be the definite integral

$$\int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx$$

It can be proved (not by us now) that the limit is indeed the definite integral even if $c_k \neq d_k$.

Surface Area Definition

Let f be a nonnegative, smooth function on $[a, b]$. Then the **surface area** S generated by revolving the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ about the x-axis is

$$S = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx$$

Let $x = g(y)$ be a nonnegative, smooth function on $[c, d]$. Then the **surface area** S generated by revolving the portion of the curve $x = g(y)$ between $y = c$ and $y = d$ about the y-axis is

$$S = \int_c^d 2\pi g(y)\sqrt{1 + (g'(y))^2}\, dy$$

Example 1

Find the surface area generated by revolving the curve

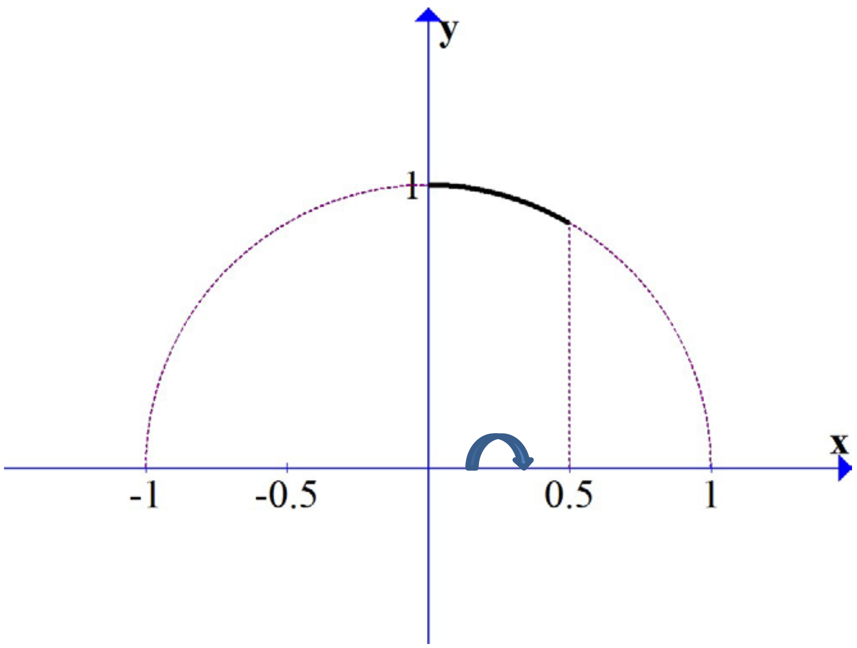
$$y = \sqrt{1 - x^2}, \qquad 0 \leq x \leq \frac{1}{2}$$

about the x-axis.

Solution:

The graph of the curve is the upper semi-circle of radius 1 centered at the origin.

Example 1 (continued)



$$y = \sqrt{1 - x^2}$$
$$\frac{dy}{dx} = \frac{-x}{\sqrt{1 - x^2}}$$

$$S = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx$$
$$= \int_0^{1/2} 2\pi\sqrt{1 - x^2}\sqrt{1 + \left(\frac{-x}{\sqrt{1 - x^2}}\right)^2}\,dx$$
$$= \int_0^{1/2} 2\pi\sqrt{1 - x^2}\sqrt{1 + \frac{x^2}{1 - x^2}}\,dx$$
$$= \int_0^{1/2} 2\pi\sqrt{1 - x^2}\sqrt{\frac{1}{1 - x^2}}\,dx$$
$$= \int_0^{1/2} 2\pi\,dx$$
$$= \cdots = \pi$$

Example 2

Find the surface area generated by revolving the curve

$$y = \sqrt[3]{3x}, \qquad 0 \le y \le 2$$

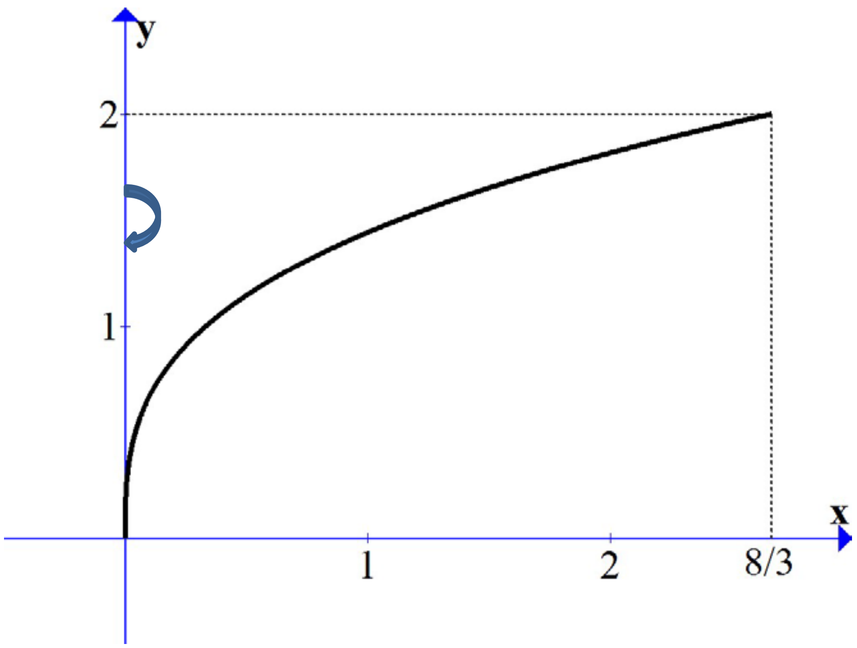
about the y-axis.

<u>Solution</u>:

$$y = \sqrt[3]{3x} \Rightarrow x = \frac{1}{3}y^3$$

$$\frac{dx}{dy} = y^2$$

Example 2 (continued)



$$S = \int_c^d 2\pi g(y)\sqrt{1 + (g'(y))^2}\, dy$$

$$= \int_0^2 2\pi \left(\frac{1}{3}y^3\right)\sqrt{1 + (y^2)^2}\, dy$$

$$= \int_0^2 \frac{2\pi}{3} y^3 \sqrt{1 + y^4}\, dy$$

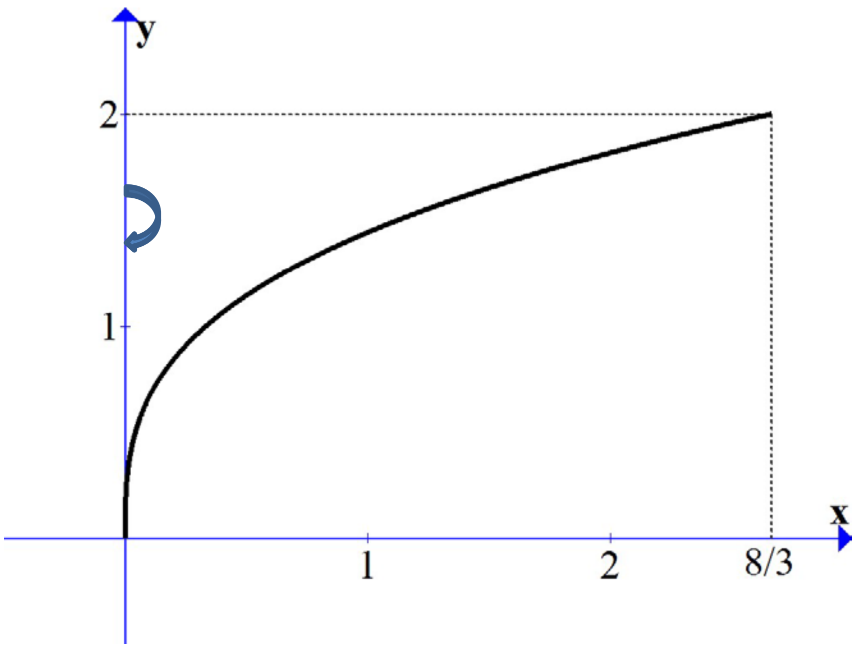
$$u = 1 + y^4$$

$$du = 4y^3 dy \Rightarrow \frac{1}{4} du = y^3 dy$$

$$y = 2 \Rightarrow u = 17$$

$$y = 0 \Rightarrow u = 1$$

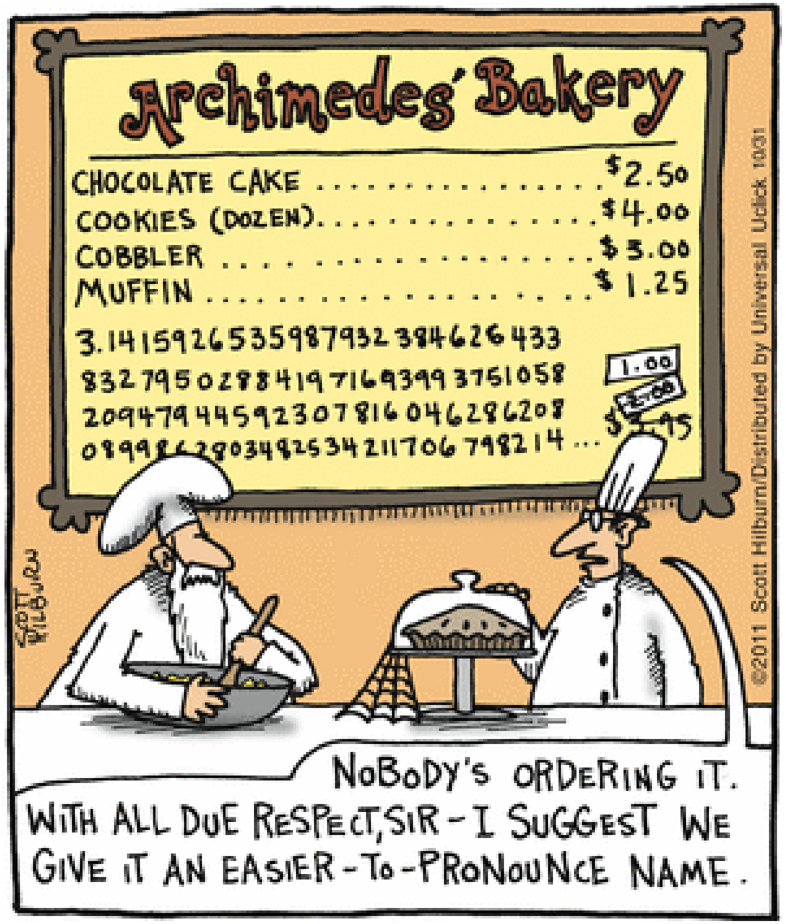
Example 2 (continued)



$$S = \int_0^2 \frac{2\pi}{3} y^3 \sqrt{1 + y^4}\, dy$$

$$= \int_1^{17} \frac{2\pi}{3} \sqrt{u} \cdot \frac{1}{4} du$$

$$= \cdots = \frac{\pi}{9}\left(17\sqrt{17} - 1\right)$$



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