Integration by Parts

Part 1

Background

If f and g are differentiable functions, then the Product Rule says:

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

Integrating both sides we get:

$$\int \frac{d}{dx}[f(x)g(x)]\,dx$$

$$= \int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx$$

Background

or

$$f(x)g(x) + C = \int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx$$

or

$$\int f(x)g'(x)\,dx = f(x)g(x) + C - \int g(x)f'(x)\,dx$$

Since the integral on the right will produce another constant of integration, there is no need to keep the C in this last equation, giving:

Background

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

Letting $u = f(x) \Rightarrow du = f'(x)dx$ and $v = g(x) \Rightarrow dv = g'(x)dx$, this becomes:

$$\int u\,dv = uv - \int v\,du$$

Integration by Parts

Integration by Parts for Indefinite Integrals

$$\int u\,dv = uv - \int v\,du$$

Integration by Parts for Definite Integrals

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

Example 1

Evaluate

$$\int xe^x\,dx.$$

Example 1 (continued) - $\int xe^x \, dx$

Solution:

Try to recognize $\int xe^x \, dx$ as $\int u \, dv$.

$u = x$	$dv = e^x dx$
$du = dx$	$v = \int e^x \, dx = e^x$

$$\int xe^x \, dx = uv - \int v \, du$$

$$= xe^x - \int e^x \, dx$$

$$= \boxed{xe^x - e^x + C}$$

Example 2

Evaluate $\int x^2 e^{-x}\,dx$.

Solution:

$u = x^2$	$dv = e^{-x}\,dx$
$du = 2x\,dx$	$v = \int e^{-x}\,dx = -e^{-x}$

$$\int x^2 e^{-x}\,dx = uv - \int v\,du$$

$$= x^2(-e^{-x}) - \int (-e^{-x})(2x\,dx\,)$$

Example 2 (continued)

$$x^2(-e^{-x}) - \int (-e^{-x})(2x\,dx) = -x^2e^{-x} + 2\int xe^{-x}\,dx$$

$u = x$	$dv = e^{-x}\,dx$
$du = dx$	$v = \int e^{-x}\,dx = -e^{-x}$

$$= -x^2e^{-x} + 2\left(uv - \int v\,du\right)$$

$$= -x^2e^{-x} + 2\left(x(-e^{-x}) - \int (-e^{-x})\,dx\right)$$

Example 2 (continued)

$$-x^2e^{-x} + 2\left(x(-e^{-x}) - \int(-e^{-x})\,dx\right)$$

$$= -x^2e^{-x} - 2xe^{-x} + 2\int e^{-x}\,dx$$

$$= -x^2e^{-x} - 2xe^{-x} + 2(-e^{-x}) + C$$

$$= -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$$

$$= \boxed{-(x^2 + 2x + 2)e^{-x} + C}$$

Example 3

Evaluate $\int \ln x \, dx$.

Solution:

$u = \ln x$	$dv = dx$
$du = \frac{1}{x} dx$	$v = \int dx = x$

$$\int \ln x \, dx = uv - \int v \, du$$

$$= (\ln x)x - \int x \left(\frac{1}{x} dx \right)$$

$$= x \ln x - \int dx$$

$$= \boxed{x \ln x - x + C}$$

Example 4

Evaluate $\int_0^1 \tan^{-1} x\, dx$.

Solution:

$u = \tan^{-1} x$	$dv = dx$
$du = \dfrac{1}{1+x^2}\, dx$	$v = x$

$$\int_0^1 \tan^{-1} x\, dx = uv\Big|_a^b - \int_a^b v\, du$$

$$= (\tan^{-1} x)x\Big|_0^1 - \int_0^1 x\left(\frac{1}{1+x^2}\, dx\right)$$

$$= [(\tan^{-1} 1)1 - (\tan^{-1} 0)0] - \int_0^1 \frac{x}{1+x^2}\, dx$$

Example 4 (continued)

$$\left[(\tan^{-1} 1)\cdot 1 - (\tan^{-1} 0)\cdot 0\right] - \int_0^1 \frac{x}{1+x^2}\,dx$$

$$= \left[\frac{\pi}{4}\cdot 1 - 0\cdot 0\right] - \int_0^1 \frac{x}{1+x^2}\,dx \;= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}\,dx$$

$$u = 1 + x^2$$

$$du = 2x\,dx \Longrightarrow \frac{1}{2}du = x\,dx$$

$$x = 1 \Longrightarrow u = 2$$

$$x = 0 \Longrightarrow u = 1$$

Example 4 (continued)

$$\int_0^1 \tan^{-1} x \, dx = \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2} dx$$

$$= \frac{\pi}{4} - \int_1^2 \frac{1}{u} \cdot \frac{1}{2} du$$

$$= \frac{\pi}{4} - \frac{1}{2} \ln|u| \Big|_1^2$$

$$= \frac{\pi}{4} - \left(\frac{1}{2} \ln 2 - \frac{1}{2} \ln 1 \right)$$

$$= \frac{\pi}{4} - \left(\ln \sqrt{2} - \frac{1}{2} \cdot 0 \right) = \frac{\pi}{4} - \ln \sqrt{2}$$

Example 5

Evaluate $\int e^x \cos x \, dx$.

Solution:

$u = e^x$	$dv = \cos x \, dx$
$du = e^x \, dx$	$v = \sin x$

$$\int e^x \cos x \, dx = uv - \int v \, du = e^x \sin x - \int \sin x \, (e^x \, dx)$$

$$= e^x \sin x - \int e^x \sin x \, dx$$

Example 5 (continued)

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$$

$u = e^x$	$dv = \sin x$
$du = e^x \, dx$	$v = -\cos x \, dx$

$$\int e^x \cos x \, dx = e^x \sin x - \left(uv - \int v \, du \right)$$

$$= e^x \sin x - \left(e^x(-\cos x) - \int (-\cos x)(e^x \, dx) \right)$$

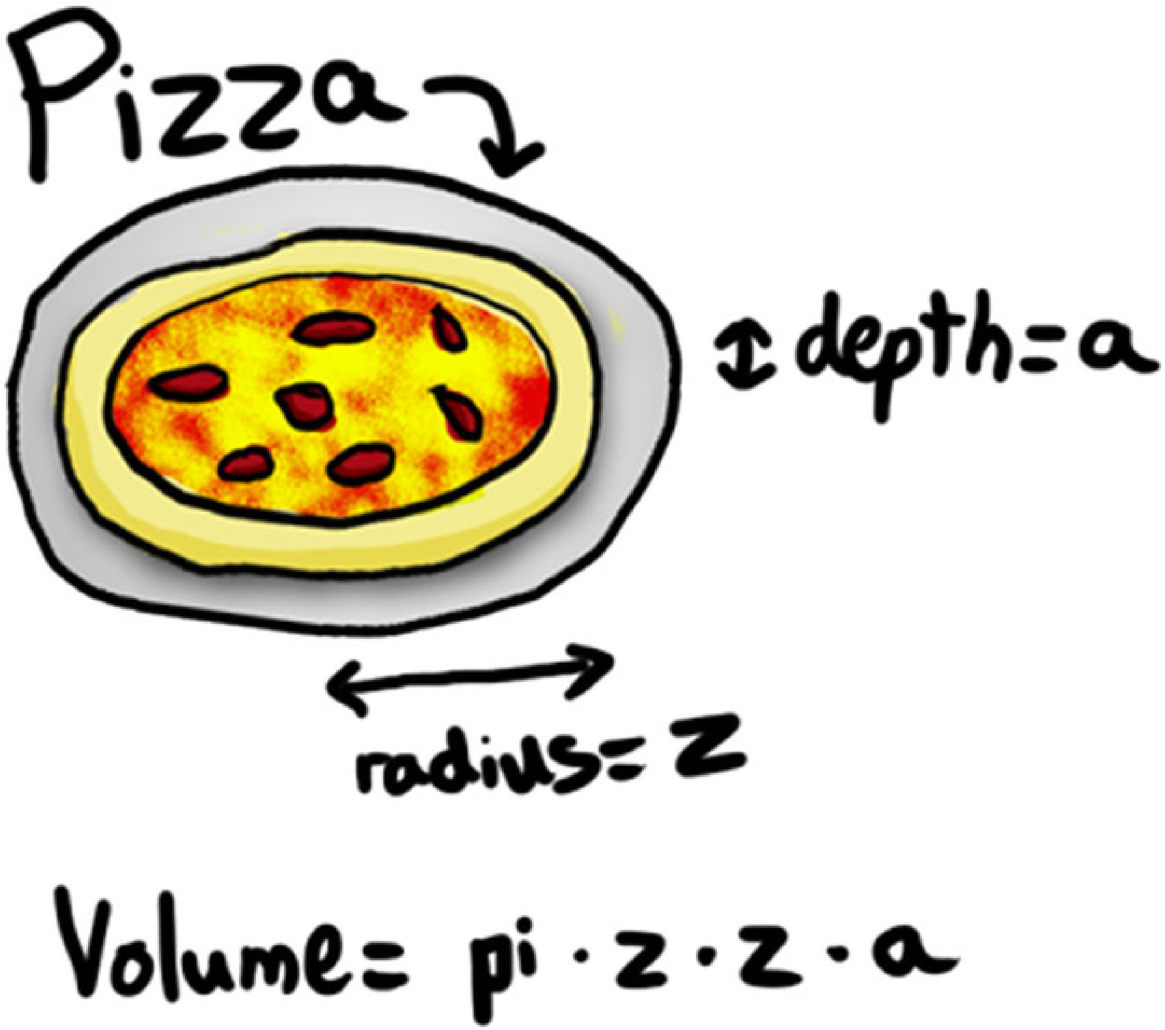
$$= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

Example 5 (continued)

$$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

$$2 \int e^x \cos x \, dx = e^x \sin x + e^x \cos x + C$$

$$\int e^x \cos x \, dx = \boxed{\frac{e^x \sin x + e^x \cos x}{2} + C}$$



http://blog.jayare.eu/pizza-math.html