Trigonometric Integrals

Part 3: Powers of Secant and Tangent

Recall

$$\int \tan x \, dx = \ln|\sec x| + C$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

$$1 + \tan^2 x = \sec^2 x$$

Example 1

Evaluate

$$\int \sec^3 x \, dx$$

<u>Solution</u>:

$$\int \sec^3 x \, dx = \int (1 + \tan^2 x) \sec x \, dx$$

$$= \int \sec x \, dx + \int \tan^2 x \sec x \, dx$$

Example 1 (continued)

$$\int \sec x \, dx + \int \tan^2 x \sec x \, dx$$

$$= \ln|\sec x + \tan x| + \int \tan x \sec x \tan x \, dx$$

$u = \tan x$	$dv = \sec x \tan x \, dx$
$du = \sec^2 x \, dx$	$v = \sec x$

$$= \ln|\sec x + \tan x| + \tan x \sec x - \int \sec x \sec^2 x \, dx$$

$$= \ln|\sec x + \tan x| + \sec x \tan x - \int \sec^3 x \, dx$$

Example 1 (continued)

$$\int \sec^3 x\,dx = \ln|\sec x + \tan x| + \sec x \tan x - \int \sec^3 x\,dx$$

$$2\int \sec^3 x\,dx = \ln|\sec x + \tan x| + \sec x \tan x + C$$

$$\int \sec^3 x\,dx = \frac{1}{2}(\ln|\sec x + \tan x| + \sec x \tan x) + C$$

$$= \boxed{\frac{1}{2}\ln|\sec x + \tan x| + \frac{1}{2}\sec x \tan x + C}$$

Integrating Power of Secant and Tangent

If m, n are non-negative integers, then

$$\int \tan^m x \sec^n x \, dx$$

can be evaluated by one of the following three procedures:

Integrating Power of Secant and Tangent

$$\int \tan^m x \sec^n x \, dx$$

Case	Procedure	Relevant Identities
n even	Substitute $u = \tan x$	$\sec^2 x = \tan^2 x + 1$
m odd	Substitute $u = \sec x$	$\tan^2 x = \sec^2 x - 1$
n odd & m even	Use identities to reduce powers of secant	$\tan^2 x = \sec^2 x - 1$

Example 2

Evaluate

$$\int \tan^2 x \sec^4 x \, dx$$

<u>Solution</u>:

Since the power of secant is even, we will "peel" off two powers of secant and re-write the remaining powers in terms of tangent by using $\sec^2 x = \tan^2 x + 1$.

Example 2 (continued)

$$\int \tan^2 x \sec^4 x \, dx = \int \tan^2 x \sec^2 x \sec^2 x \, dx$$

$$= \int \tan^2 x \left(\tan^2 x + 1\right) \sec^2 x \, dx$$

$$\boxed{\begin{gathered} u = \tan x \\ du = \sec^2 x \, dx \end{gathered}}$$

Example 2 (continued)

$$\int \tan^2 x\,(\tan^2 x + 1)\sec^2 x\,dx = \int u^2(u^2+1)\,du$$

$$= \int (u^4 + u^2)\,du$$

$$= \frac{1}{5}u^5 + \frac{1}{3}u^3 + C$$

$$= \boxed{\frac{1}{5}\tan^5 x + \frac{1}{3}\tan^3 x + C}$$

Example 3

Evaluate

$$\int \tan^3 x \sec^3 x \, dx$$

<u>Solution</u>:

Since the power of tangent is odd, we will "peel" off a power of secant and re-write the even powers of tangent in terms of secant by using $\tan^2 x = \sec^2 x - 1$.

Example 3 (continued)

$$\int \tan^3 x \sec^3 x \, dx = \int \tan^2 x \sec^2 x \sec x \tan x \, dx$$

$$= \int (\sec^2 x - 1) \sec^2 x \sec x \tan x \, dx$$

$$u = \sec x$$
$$du = \sec x \tan x \, dx$$

$$= \int (u^2 - 1) u^2 \, du$$

Example 3 (continued)

$$\int (u^2 - 1)u^2\, du = \int (u^4 - u^2)\, du$$

$$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C$$

$$= \boxed{\frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C}$$

Example 4

Evaluate

$$\int \tan^2 x \sec x \, dx$$

<u>Solution</u>:

Since the power tangent is even and the power of secant is odd, we will re-write the even powers of tangent in terms of secant by using $\tan^2 x = \sec^2 x - 1$.

Example 4 (continued)

$$\int \tan^2 x \sec x \, dx = \int (\sec^2 x - 1) \sec x \, dx$$

$$= \int \sec^3 x \, dx - \int \sec x \, dx$$

$$= \int \sec^3 x \, dx - \ln|\sec x + \tan x|$$

Example 4 (continued)

From Example 1, we know:

$$\int \sec^3 x\,dx = \frac{1}{2}\ln|\sec x + \tan x| + \frac{1}{2}\sec x \tan x + C$$

So we get

$$\int \tan^2 x \sec x\,dx$$

$$= \frac{1}{2}\ln|\sec x + \tan x| + \frac{1}{2}\sec x \tan x - \ln|\sec x + \tan x| + C$$

$$= \boxed{\frac{1}{2}\sec x \tan x - \frac{1}{2}\ln|\sec x + \tan x| + C}$$

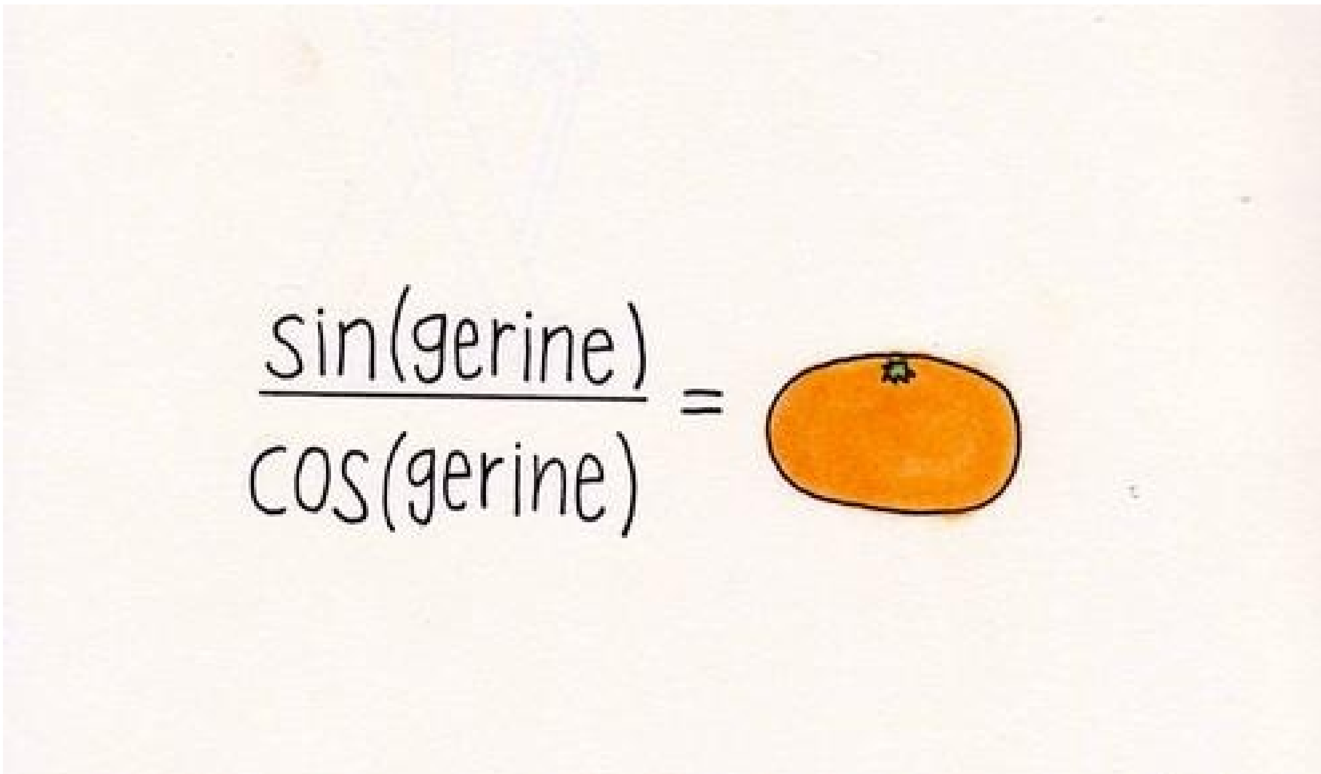
Closing Remark

By using

$$1 + \cot^2 x = \csc^2 x$$

you can adapt these techniques to solve

$$\int \cot^m x \csc^n x \, dx$$



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