Trigonometric Substitutions

Part 3:

$\sqrt{x^2 - a^2}$ and Procedures

Example

Evaluate

$$\int \frac{\sqrt{x^2 - 25}\, dx}{x}$$

<u>Solution</u>:

This has an expression involving the form

$$\sqrt{x^2 - a^2}$$

where $a = 5$.

Example (continued)

We need x to take on every value less than or equal to -5 and greater than or equal to 5.

Since $\sec\theta$ takes on every value less than or equal to -1 and greater than or equal to 1 exactly once on $0 \leq \theta \leq \pi, \theta \neq \frac{\pi}{2}$, we will let

$$x = 5\sec\theta\,, \qquad 0 \leq \theta \leq \pi, \theta \neq \frac{\pi}{2}$$

Now, let's set up our substitutions and our reference triangle.

Example (continued)

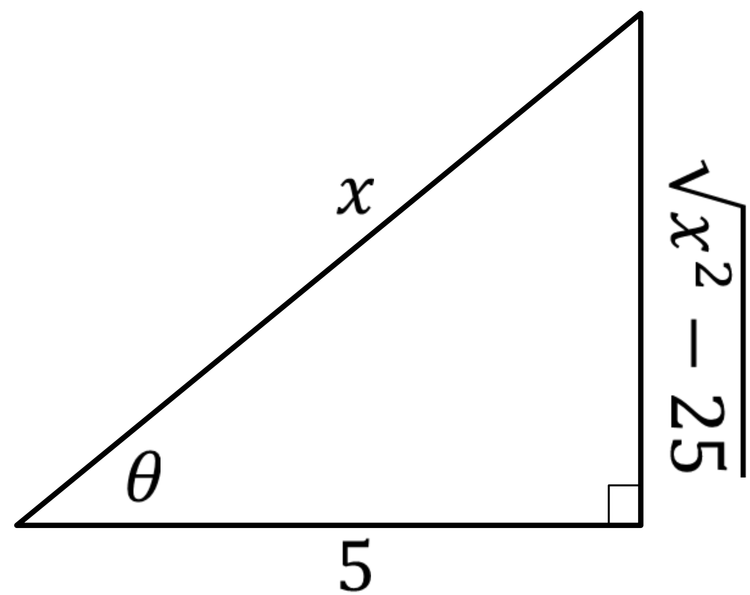
$$x = 5\sec\theta\,,$$

$$0 \le \theta \le \pi, \theta \ne \frac{\pi}{2}$$

$$dx = 5\sec\theta\tan\theta\,d\theta$$

$$\frac{x}{5} = \sec\theta = \frac{\text{HYP}}{\text{ADJ}}$$

$$\Rightarrow \sec^{-1}\left(\frac{x}{5}\right) = \theta$$



Example (continued)

$$\int \frac{\sqrt{x^2 - 25}\,dx}{x} = \int \frac{\sqrt{(5\sec\theta)^2 - 25} \cdot 5\sec\theta\tan\theta\,d\theta}{5\sec\theta}$$

$$= \int \sqrt{25\sec^2\theta - 25}\tan\theta\,d\theta$$

$$= \int \sqrt{25(\sec^2\theta - 1)}\tan\theta\,d\theta$$

$$= \int \sqrt{25\tan^2\theta}\tan\theta\,d\theta$$

Example (continued)

$$\int \sqrt{25\tan^2\theta}\,\tan\theta\,d\theta = \int 5\tan\theta\tan\theta\,d\theta$$

$$= \int 5\tan^2\theta\,d\theta$$

$$= 5\int (\sec^2\theta - 1)\,d\theta$$

$$= 5(\tan\theta - \theta) + C$$

Now, going back to our reference triangle, we see

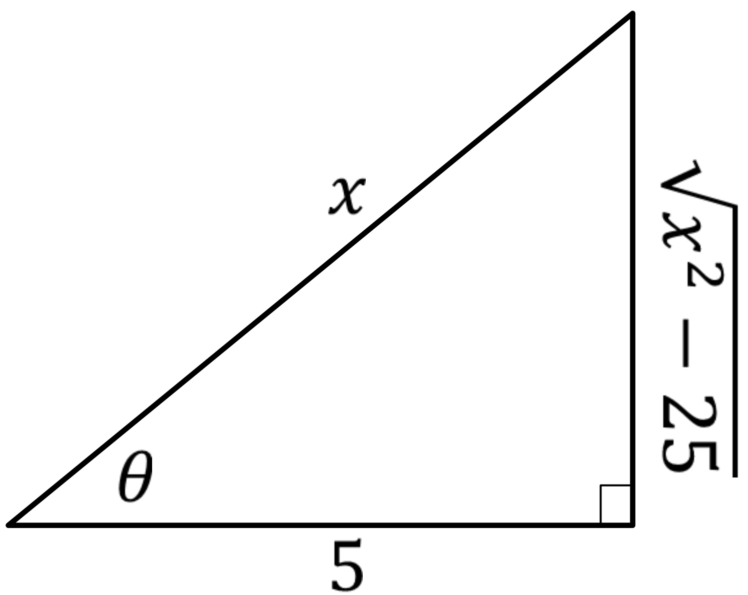
Example (continued)

$$x = 5\sec\theta\,,$$
$$0 \le \theta \le \pi, \theta \ne \frac{\pi}{2}$$

$$\sec\theta = \frac{\text{HYP}}{\text{ADJ}} = \frac{x}{5}$$

$$\Rightarrow \sec^{-1}\left(\frac{x}{5}\right) = \theta$$

$$\tan\theta = \frac{\text{OPP}}{\text{ADJ}} = \frac{\sqrt{x^2 - 25}}{5}$$



Example (continued)

This gives

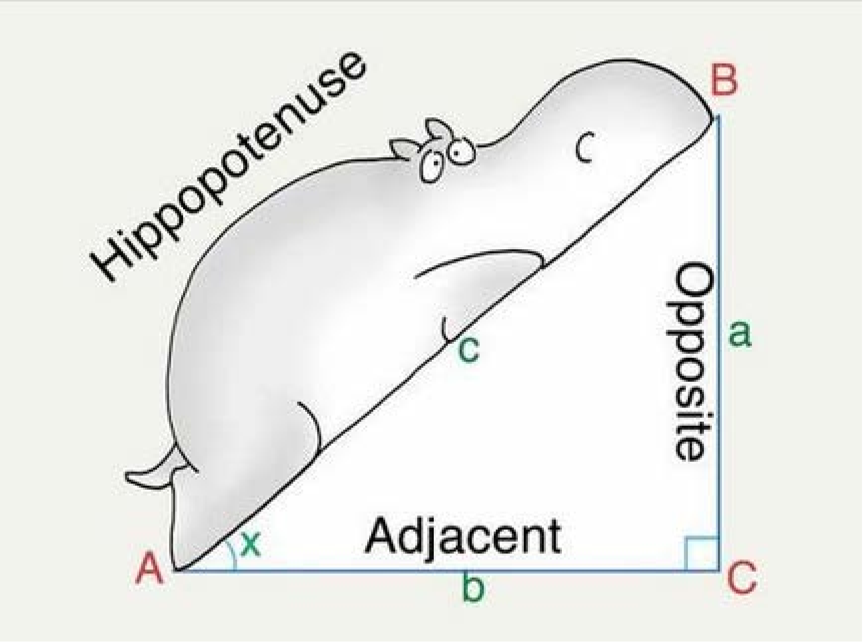
$$\int \frac{\sqrt{x^2 - 25}\,dx}{x} = 5(\tan\theta - \theta) + C$$

$$= 5\left(\frac{\sqrt{x^2 - 25}}{5} - \sec^{-1}\left(\frac{x}{5}\right)\right) + C$$

$$= \boxed{\sqrt{x^2 - 25} - 5\sec^{-1}\left(\frac{x}{5}\right) + C}$$

Procedure

- Make the appropriate substitutions:
  - $\sqrt{a^2 - x^2}$: $x = a \sin\theta$ & $dx = a\cos\theta\, d\theta$
  - $\sqrt{x^2 + a^2}$: $x = a\tan\theta$ & $dx = a\sec^2\theta\, d\theta$
  - $\sqrt{x^2 - a^2}$: $x = a\sec\theta$ & $dx = a\sec\theta\tan\theta\, d\theta$
- Draw a reference triangle
- Evaluate the integral (which is now in terms of θ)
- Use the reference triangle to re-write the answer in terms of x



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