Integration of Rational Functions by Partial Fractions

Part 1:

Integrals involving $ax^2 + bx + c$

Completing the Square

Goal: Rewrite $ax^2 + bx + c$ in the form $a(x-h)^2 + k$

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x\right) + c$$

$$= a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right) + c - a\left(\frac{b}{2a}\right)^2$$

$$= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

$$h = -\frac{b}{2a}, k = c - \frac{b^2}{4a}$$

Integrals involving $ax^2 + bx + c$

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

When you integrate, make the substitution

$$u = x + \frac{b}{2a}$$

so that

$$ax^2 + bx + c = a \cdot u^2 + d$$

where

$$d = c - \frac{b^2}{4a}$$

Example 1

Evaluate

$$\int \frac{1}{x^2 - 2x + 5} dx$$

<u>Solution</u>:

Completing the square we get:

$$x^2 - 2x + 5 = (x^2 - 2x + 1) + 5 - 1$$
$$= (x-1)^2 + 4$$

Example 1 (continued)

$$\int \frac{1}{x^2 - 2x + 5} dx = \int \frac{1}{(x-1)^2 + 4} dx$$

$$u = x - 1$$
$$du = dx$$

$$= \int \frac{1}{u^2 + 4} du$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}$$

$$= \int \frac{1}{4\left(\frac{u^2}{4} + 1\right)} du$$

Example 1 (continued)

$$\int \frac{1}{4\left(\frac{u^2}{4}+1\right)}du = \frac{1}{4}\int \frac{1}{\left(\frac{u}{2}\right)^2+1}du$$

$$= \frac{1}{4}\cdot 2\tan^{-1}\left(\frac{u}{2}\right)+C$$

$$= \boxed{\frac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right)+C}$$

Example 2

Evaluate

$$\int \frac{x}{x^2 - 4x + 8}\, dx$$

<u>Solution</u>:

Completing the square we get:

$$x^2 - 4x + 8 = (x^2 - 4x + 4) + 8 - 4$$

$$= (x - 2)^2 + 4$$

Example 2 (continued)

$$\int \frac{x}{x^2 - 4x + 8} dx = \int \frac{x}{(x-2)^2 + 4} dx$$

$$u = x - 2 \Rightarrow u + 2 = x$$
$$du = dx$$

$$= \int \frac{u + 2}{u^2 + 4} du$$

$$= \int \frac{u}{u^2 + 4} du + \int \frac{2}{u^2 + 4} du$$

Example 2 (continued)

From Example 1, we know that

$$\int \frac{1}{u^2+4}\,du = \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right) + C$$

Using this we get

$$\int \frac{u}{u^2+4}\,du + \int \frac{2}{u^2+4}\,du$$

$$= \int \frac{u}{u^2+4}\,du + 2 \cdot \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right)$$

Example 2 (continued)

$$\int \frac{u}{u^2+4}\,du + 2\cdot\frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right)$$

$$v = u^2 + 4$$

$$dv = 2u\,du \Rightarrow \frac{1}{2}dv = u\,du$$

$$= \int \frac{1}{v}\cdot\frac{1}{2}dv + \tan^{-1}\left(\frac{u}{2}\right)$$

$$= \frac{1}{2}\cdot\ln|v| + \tan^{-1}\left(\frac{u}{2}\right) + C$$

Example 2 (continued)

$$\frac{1}{2}\cdot\ln|v|+\tan^{-1}\left(\frac{u}{2}\right)+C$$

$$=\frac{1}{2}\cdot\ln|u^2+4\,|+\tan^{-1}\left(\frac{u}{2}\right)+C$$

$$=\boxed{\frac{1}{2}\cdot\ln\left((x-2)^2+4\right)+\tan^{-1}\left(\frac{x-2}{2}\right)+C}$$

Example 3

Evaluate

$$\int \frac{1}{\sqrt{5-4x-2x^2}}\,dx$$

<u>Solution</u>:

Completing the square we get:

$$-2x^2-4x+5=-2(x^2+2x+1)+5+2$$
$$=-2(x+1)^2+7$$

Example 3 (continued)

$$\int \frac{1}{\sqrt{5-4x-2x^2}}\,dx = \int \frac{1}{\sqrt{-2(x+1)^2+7}}\,dx$$

$$\boxed{\begin{matrix} u = x+1 \\ du = dx \end{matrix}}$$

$$= \int \frac{1}{\sqrt{-2u^2+7}}\,du = \int \frac{1}{\sqrt{7-2u^2}}\,du$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

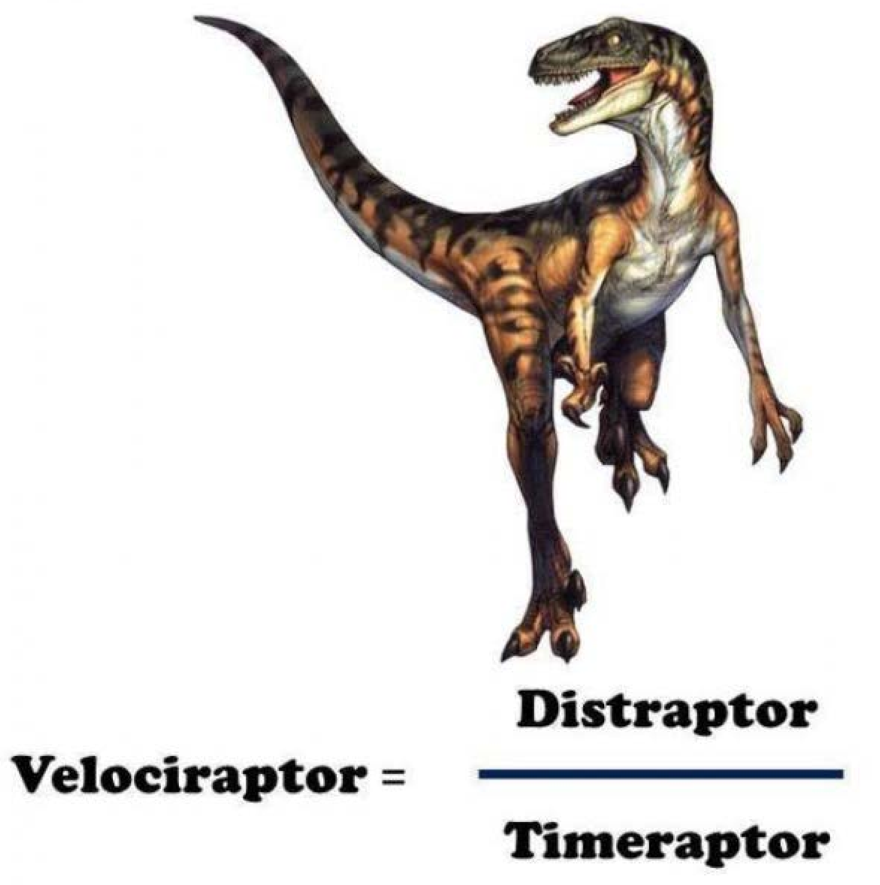
$$= \int \frac{1}{\sqrt{7\left(1-\frac{2}{7}u^2\right)}}\,du$$

Example 3 (continued)

$$\int \frac{1}{\sqrt{7\left(1-\frac{2}{7}u^2\right)}}du = \int \frac{1}{\sqrt{7}\sqrt{1-\left(\sqrt{\frac{2}{7}}\cdot u\right)^2}}du$$

$$= \frac{1}{\sqrt{7}} \cdot \sqrt{\frac{7}{2}} \cdot \sin^{-1}\left(\sqrt{\frac{2}{7}} \cdot u\right) + C$$

$$= \frac{1}{\sqrt{2}} \cdot \sin^{-1}\left(\sqrt{\frac{2}{7}} \cdot (x+1)\right) + C$$



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