Integration of Rational Functions by Partial Fractions

Part 2:

Integrating Rational Functions

Rational Functions

Recall that a **rational function** is the quotient of two polynomials.

$$\frac{1}{x} + \frac{3}{x-1} + \frac{2}{x+2}$$
$$= \frac{1(x-1)(x+2) + 3x(x+2) + 2x(x-1)}{x(x-1)(x+2)}$$
$$= \frac{6x^2 + 5x - 2}{x^3 + x^2 - 2x}$$

The left side of this equation is easy to integrate.

The right side is hard to integrate.

Rational Functions

Suppose $p(x)$ and $q(x)$ are polynomials and

$$f(x) = \frac{p(x)}{q(x)}.$$

- If the degree of $p(x)$ is less than the degree of $q(x)$, then $f(x)$ is a **proper** rational function.
- If the degree of $p(x)$ is greater than or equal to the degree of $q(x)$, then $f(x)$ is an **improper** rational function.

Partial Fraction Decomposition

In theory, a polynomial with real coefficients can always be factored into a product of linear and quadratic factors.

If a quadratic factor cannot be further decomposed into linear factors, then it is said to be **irreducible**.

It can be proved that any proper rational function is expressible as a sum of terms (called **partial fractions**) having the form:

$$\frac{A}{(ax+b)^k} \text{ or } \frac{Bx+C}{(ax^2+bx+c)^k}.$$

Steps to Partial Fraction Decomposition

Suppose $p(x)$ and $q(x)$ are polynomials and

$$f(x) = \frac{p(x)}{q(x)}.$$

- Completely factor the denominator $q(x)$ into linear and irreducible quadratic factors.
- Collect all repeated factors so that $q(x)$ is expressed as a product of *distinct* factors of the form

  $$(ax + b)^m \text{ and } (ax^2 + bx + c)^m$$

  where $(ax^2 + bx + c)^m$ is irreducible.

Steps to Partial Fraction Decomposition (continued)

- The structure of $\frac{p(x)}{q(x)}$ is determined as follows:

Linear Factors

For each factor of the form $(ax+b)^m$, introduce the m terms

$$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_m}{(ax+b)^m}$$

where $A_1, A_2, \cdots, A_m$ are constants to be determined.

Steps to Partial Fraction Decomposition (continued)

Irreducible Quadratic Factors

For each factor of the form $(ax^2 + bx + c)^m$, introduce the m terms

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_mx + B_m}{(ax^2 + bx + c)^m}$$

where $A_1, A_2, \cdots, A_m, B_1, B_2, \cdots, B_m$ are constants to be determined.

Example 1

(1) $\frac{1}{(x-1)^2(x+3)^3(x^2+x+1)^2}$

$$= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+3} + \frac{D}{(x+3)^2} + \frac{E}{(x+3)^3}$$
$$+ \frac{Fx+G}{x^2+x+1} + \frac{Hx+I}{(x^2+x+1)^2}$$

(2) $\frac{5x+4}{x^2(x^2+4)}$

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+4}$$

Example 2

Evaluate

$$\int \frac{1}{x^2 + x - 2} dx$$

<u>Solution</u>:

$$\frac{1}{x^2 + x - 2} = \frac{1}{(x-1)(x+2)}$$

Example 2 (continued)

$$\frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

Now, multiply both sides of the equation by $(x-1)(x+2)$ to get

$$1 = A(x+2) + B(x-1)$$

To solve for A and B, substitute values of x to make the various terms zero.

Example 2 (continued)

$$1 = A(x + 2) + B(x - 1)$$

Setting $x = -2$ gives us:

$$1 = A(-2 + 2) + B(-2 - 1)$$

$$1 = -3B$$

$$-\frac{1}{3} = B$$

Setting $x = 1$ gives us:

$$1 = A(1 + 2) + B(1 - 1)$$

$$1 = 3A$$

$$\frac{1}{3} = A$$

Example 2 (continued)

An alternate way to solve for A and B is:

Take the equation

$$1 = A(x+2) + B(x-1)$$

Equate corresponding coefficients on both sides

$$1 = Ax + 2A + Bx - B$$

$$0x + 1 = (A+B)x + (2A - B)$$

$$\begin{cases} A + B = 0 \\ 2A - B = 1 \end{cases}$$

Solve the system of equations to get

$$A = \frac{1}{3} \text{ and } B = -\frac{1}{3}.$$

Example 2 (continued)

Since

$$\frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$
$$= \frac{1/3}{x-1} + \frac{-1/3}{x+2}$$

we can now say

$$\int \frac{1}{(x-1)(x+2)}\,dx = \frac{1}{3}\int \frac{1}{x-1}\,dx - \frac{1}{3}\int \frac{1}{x+2}\,dx$$
$$= \frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2| + C$$
$$= \boxed{\frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + C}$$

Example 3

Evaluate

$$\int \frac{3x^4 + 3x^3 - 5x^2 + x - 1}{x^2 + x - 2}\,dx$$

<u>Solution</u>:

Since the degree of the numerator is larger than the degree of the denominator, this is an improper rational function!

Example 3 (continued)

When you have an improper rational function, the first thing you need to do is long division of polynomials to rewrite the improper rational function as the sum of a polynomial and a proper rational function.

Using long division of polynomials we get:

$$\frac{3x^4 + 3x^3 - 5x^2 + x - 1}{x^2 + x - 2}$$

$$= (3x^2 + 1) + \frac{1}{x^2 + x - 2}$$

Example 3 (continued)

$$\int \frac{3x^4 + 3x^3 - 5x^2 + x - 1}{x^2 + x - 2} dx$$

$$= \int (3x^2 + 1)\, dx + \underbrace{\int \frac{1}{x^2 + x - 2} dx}_{\text{See Example 2}}$$

$$= \boxed{x^3 + x + \frac{1}{3} \ln \left| \frac{x-1}{x+2} \right| + C}$$

http://math-fail.com/2011/12/asymptotic-high-fives.html

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