Miscellaneous Substitutions

Example 1
(Rational Powers of x)

Evaluate

$$\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,dx$$

<u>Solution</u>:

Let n be the least common multiple of the denominators of the exponents of x.

$$n = \text{LCM}(2,3) = 6$$

Example 1 (continued)

Now substitute $u = x^{1/n}$:

$$u = x^{1/6} \Rightarrow u^6 = x$$

$$du = \frac{1}{6} x^{-5/6}\, dx \Rightarrow 6u^5\, du = dx$$

$$\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\, dx = \int \frac{\sqrt{u^6}}{1+\sqrt[3]{u^6}} \cdot 6u^5\, du$$

$$= \int \frac{u^3}{1+u^2} \cdot 6u^5\, du$$

$$= 6 \int \frac{u^8}{1+u^2}\, du$$

Example 1 (continued)

Now, using long division of polynomials we can find:

$$\frac{u^8}{1+u^2} = u^6 - u^4 + u^2 - 1 + \frac{1}{1+u^2}$$

Using this we get:

$$6\int \frac{u^8}{1+u^2}\,du = 6\int \left(u^6 - u^4 + u^2 - 1 + \frac{1}{1+u^2}\right)du$$

$$= 6\left(\frac{1}{7}u^7 - \frac{1}{5}u^5 + \frac{1}{3}u^3 - u + \tan^{-1} u\right) + C$$

$$= 6\left(\frac{1}{7}\left(x^{1/6}\right)^7 - \frac{1}{5}\left(x^{1/6}\right)^5 + \frac{1}{3}\left(x^{1/6}\right)^3 - \left(x^{1/6}\right) + \tan^{-1}\left(x^{1/6}\right)\right) + C$$

$$= \boxed{\frac{6}{7}x^{7/6} - \frac{6}{5}x^{5/6} + 2x^{1/2} - 6x^{1/6} + 6\tan^{-1}\left(x^{1/6}\right) + C}$$

Example 2

Evaluate

$$\int \sqrt{1+e^x}\,dx$$

<u>Solution</u>:

$$u = 1 + e^x \Rightarrow u - 1 = e^x$$

$$du = e^x\,dx \Rightarrow \frac{1}{e^x}du = \frac{1}{u-1}du = dx$$

Example 2 (continued)

$$\int \sqrt{1+e^x}\,dx = \int \frac{\sqrt{u}}{u-1}\,du$$

$$v = u^{1/2} \Rightarrow v^2 = u$$
$$2v\,dv = du$$

$$= \int \frac{v}{v^2-1} \cdot 2v\,dv$$

$$= \int \frac{2v^2}{v^2-1}\,dv$$

Example 2 (continued)

Now, using long division of polynomials we can find:

$$\frac{2v^2}{v^2-1} = 2\left(1+\frac{1}{v^2-1}\right)$$

Using this we get:

$$\int \frac{2v^2}{v^2-1}dv = 2\int\left(1+\frac{1}{v^2-1}\right)dv$$

Example 2 (continued)

Now, using partial fractions we can find:

$$\frac{1}{v^2-1} = \frac{1/2}{v-1} - \frac{1/2}{v+1}$$

Using this we get:

$$2\int\left(1+\frac{1}{v^2-1}\right)dv = 2\int\left(1+\frac{1/2}{v-1}-\frac{1/2}{v+1}\right)dv$$

$$= 2\left(v+\frac{1}{2}\ln|v-1| - \frac{1}{2}\ln|v+1|\right)+C$$

$$= 2v + \ln\left|\frac{v-1}{v+1}\right| + C$$

Example 2 (continued)

$$2v + \ln\left|\frac{v-1}{v+1}\right| + C = 2u^{1/2} + \ln\left|\frac{u^{1/2}-1}{u^{1/2}+1}\right| + C$$

$$= 2(1+e^{x})^{1/2} + \ln\left|\frac{(1+e^{x})^{1/2}-1}{(1+e^{x})^{1/2}+1}\right| + C$$

$$= \boxed{2\sqrt{1+e^{x}} + \ln\left|\frac{\sqrt{1+e^{x}}-1}{\sqrt{1+e^{x}}+1}\right| + C}$$

Example 3
(Rational Expression in $\sin x$ and $\cos x$)

Evaluate

$$\int \frac{1}{1+\sin x}\,dx$$

<u>Solution</u>:

Make the substitution:

$$u = \tan\left(\frac{x}{2}\right), \qquad -\pi < x < \pi$$

Example 3 (continued)

$$u = \tan\left(\frac{x}{2}\right), \qquad -\pi < x < \pi$$

$$\cos\left(\frac{x}{2}\right) = \frac{1}{\sec\left(\frac{x}{2}\right)} = \frac{1}{\sqrt{1 + \tan^2\left(\frac{x}{2}\right)}} = \frac{1}{\sqrt{1 + u^2}}$$

$$\sin\left(\frac{x}{2}\right) = \tan\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) = \frac{u}{\sqrt{1 + u^2}}$$

This is good, but we need $\sin x$ and $\cos x$.

Example 3 (continued)

$$\sin x = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$$

$$= 2\frac{u}{\sqrt{1+u^2}} \cdot \frac{1}{\sqrt{1+u^2}} = \frac{2u}{1+u^2}$$

$$\cos x = 1 - 2\sin^2\left(\frac{x}{2}\right)$$

$$= 1 - \left(\frac{u}{\sqrt{1+u^2}}\right)^2$$

$$= 1 - \frac{u^2}{1+u^2} = \frac{1-u^2}{1+u^2}$$

Example 3 (continued)

$$\int \frac{1}{1+\sin x}\,dx$$

$$u = \tan\left(\frac{x}{2}\right) \Rightarrow \tan^{-1} u = \frac{x}{2}$$

$$\frac{1}{1+u^2}\,du = \frac{1}{2}\,dx \Rightarrow \frac{2}{1+u^2}\,du = dx$$

$$\sin x = \frac{2u}{1+u^2}$$

Example 3 (continued)

$$\int \frac{1}{1+\sin x}\,dx = \int \frac{1}{1+\frac{2u}{1+u^2}} \cdot \frac{2}{1+u^2}\,du$$

$$= 2\int \frac{1}{(1+u^2)+2u}\,du = 2\int \frac{1}{u^2+2u+1}\,du$$

$$= 2\int \frac{1}{(u+1)^2}\,du$$

$$= 2(-(u+1)^{-1}) + C = \frac{-2}{u+1} + C$$

$$= \boxed{\frac{-2}{\tan\left(\frac{x}{2}\right)+1} + C}$$

The Reason Why Math Is Awesome

http://math-fail.com/page/28

$$1 \times 8 + 1 = 9$$
$$12 \times 8 + 2 = 98$$
$$123 \times 8 + 3 = 987$$
$$1234 \times 8 + 4 = 9876$$
$$12345 \times 8 + 5 = 98765$$
$$123456 \times 8 + 6 = 987654$$
$$1234567 \times 8 + 7 = 9876543$$
$$12345678 \times 8 + 8 = 98765432$$
$$123456789 \times 8 + 9 = 987654321$$