Numerical Integration

Part 1:

The Trapezoid Rule

Nonelementary Integrals…

…are integrals of functions whose antiderivatives cannot be expressed as finite combinations of elementary functions.

$$\int \sin x^2 \, dx, \qquad \int \sqrt{1 + x^4} \, dx, \qquad \int \frac{e^x}{x} \, dx,$$

$$\int e^{(e^x)} \, dx, \qquad \int \frac{1}{\ln x} \, dx, \qquad \int \ln(\ln x) \, dx,$$

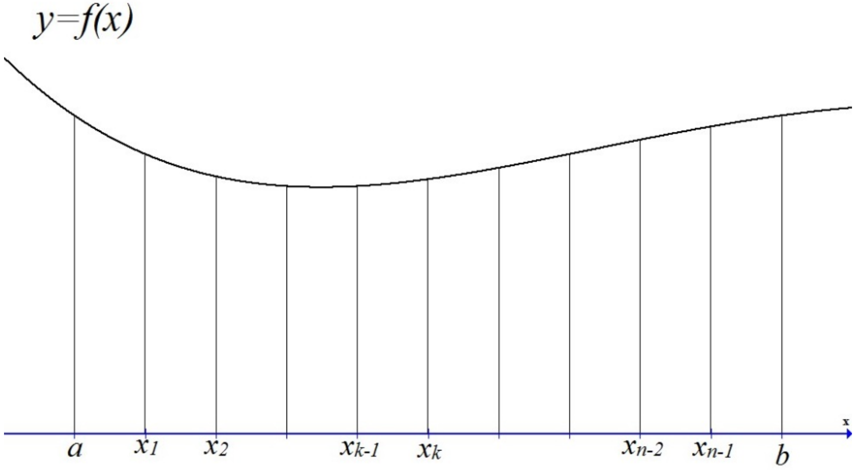
$$\int \frac{\sin x}{x} \, dx, \qquad \int \sqrt{1 - k^2 \sin^2 x} \, dx$$

Areas Using Trapezoids

Instead of using rectangles to approximate the area under the curve, we will use trapezoids.

If we subdivide the area in our normal fashion,

we can see that each slice looks like a trapezoid.



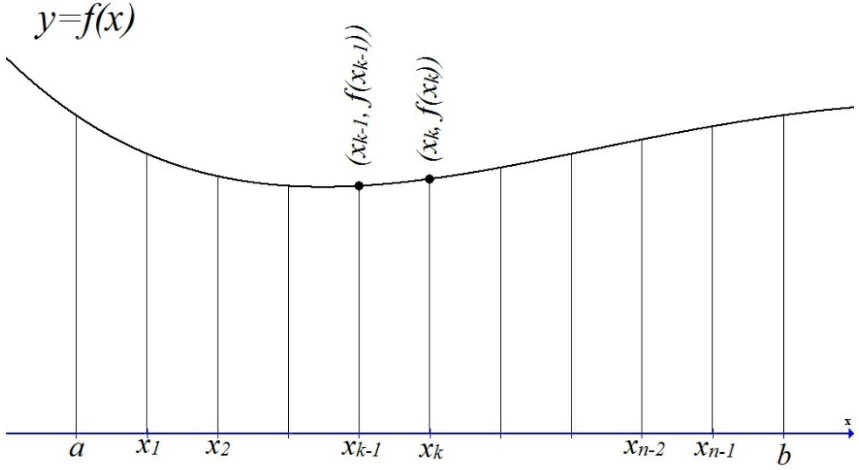
Areas Using Trapezoids

The area of the k-th slice is approximately

$$\frac{1}{2}(\text{height})(\text{sum of the bases})$$

$$= \frac{1}{2}\Delta x_k\left(f(x_{k-1}) + f(x_k)\right)$$

$$= \frac{1}{2}\Delta x_k(y_{k-1} + y_k)$$



Areas Using Trapezoids

To make our calculations easier, we will let

$$\Delta x_k = \Delta x = \frac{b-a}{n} \text{ for all } k$$

Therefore, the area under the curve is approximately equal to

$$\frac{1}{2}\Delta x(y_0 + y_1) + \frac{1}{2}\Delta x(y_1 + y_2) + \cdots + \frac{1}{2}\Delta x(y_{n-1} + y_n)$$

$$= \frac{1}{2}\Delta x[(y_0 + y_1) + (y_1 + y_2) + \cdots + (y_{n-1} + y_n)]$$

$$= \frac{\Delta x}{2}[y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n]$$

The Trapezoid Rule

To approximate $\int_a^b f(x)\,dx$, use

$$T = \frac{\Delta x}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n).$$

where

$$\Delta x = \frac{b-a}{n}$$

$$x_k = a + k \cdot \Delta x \text{ (the partition points)}$$

and

$$y_k = f(x_k).$$

Example

Approximate

$$\int_{1}^{2} \frac{1}{x} dx$$

by using the Trapezoid Rule with $n = 10$.

Example (continued)

Solution:

$$\int_1^2 \frac{1}{x}\,dx$$

$$f(x) = \frac{1}{x}$$

$$[a,b] = [1,2]$$

$$n = 10$$

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{10} = \frac{1}{10}$$

$$x_k = a + k \cdot \Delta x = 1 + \frac{k}{10}$$

$$y_k = f(x_k) = \frac{1}{x_k}$$

$$= \frac{1}{1+\frac{k}{10}} = \frac{10}{10+k}$$

Example (continued)

$$\int_1^2 \frac{1}{x}\,dx \approx \frac{\Delta x}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n)$$

$$= \frac{1/10}{2}\left(\frac{10}{10+0} + 2\cdot\frac{10}{10+1} + 2\cdot\frac{10}{10+2} + 2\cdot\frac{10}{10+3} + 2\cdot\frac{10}{10+4} + 2\cdot\frac{10}{10+5} + 2\cdot\frac{10}{10+6} + 2\cdot\frac{10}{10+7} + 2\cdot\frac{10}{10+8} + 2\cdot\frac{10}{10+9} + \frac{10}{10+10}\right)$$

$$= \frac{1}{20}\left(1 + \frac{20}{11} + \frac{20}{12} + \frac{20}{13} + \frac{20}{14} + \frac{20}{15} + \frac{20}{16} + \frac{20}{17} + \frac{20}{18} + \frac{20}{19} + \frac{1}{2}\right)$$

$$\approx 0.69377$$

(Note: $\int_1^2 \frac{1}{x}\,dx = \ln 2 - \ln 1 = \ln 2 \approx .69315$)

http://resources.mrjaffesclass.com/humor/mathville/mathville.htm