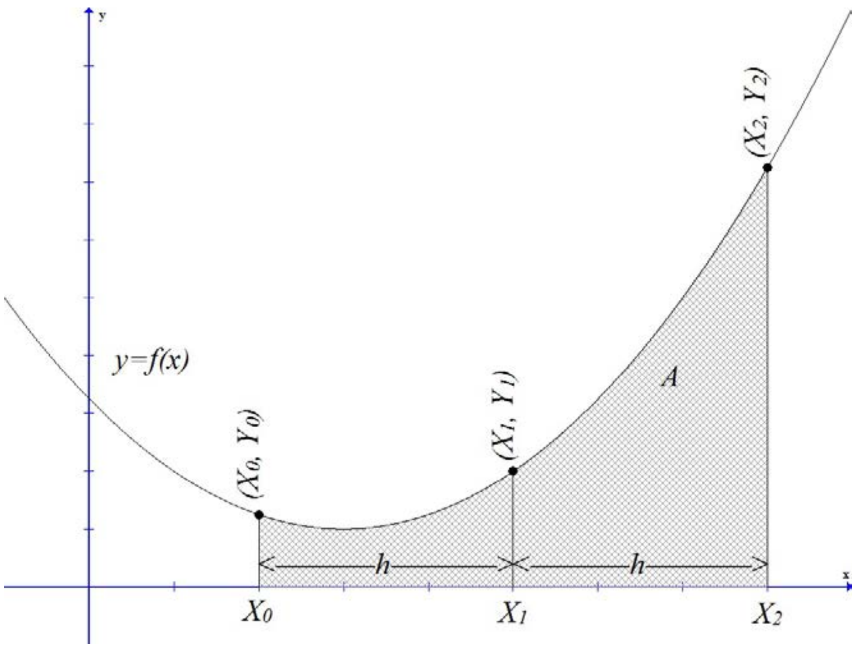
Numerical Integration

Part 2:

Simpson's Rule

Areas Using Parabolas



It can be shown that the area, A, under the curve of a parabola, $y = f(x)$, on the interval $[X_0, X_2]$ of width $2h$ is equal to

$$A = \frac{h}{3}(Y_0 + 4Y_1 + Y_2)$$

where X_1 is the midpoint of the interval and

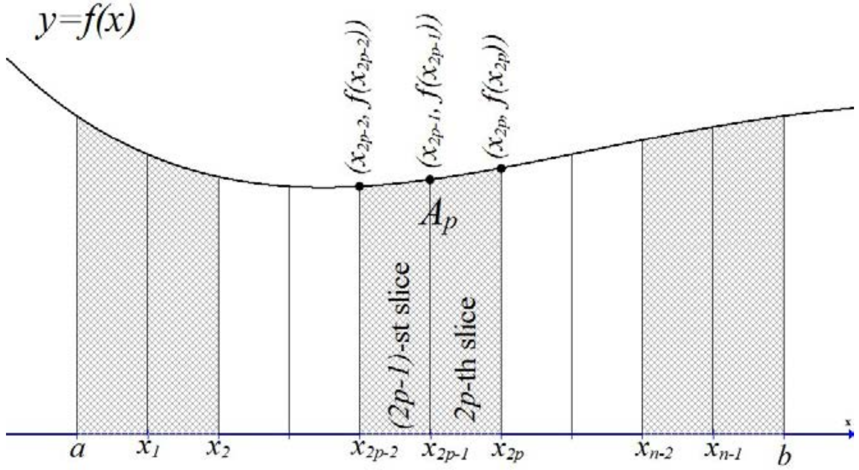
$$Y_k = f(X_k).$$

Areas Using Parabolas

Let $y = f(x)$ be a continuous function on $[a, b]$.

Divide $[a, b]$ into an *even* number, n, of subintervals of width $h = \frac{b-a}{n}$.

Let A_p represent the area of the $(2p-1)$-st slice plus the area of the $2p$-th slice. (So, $p = 1,2,3,\cdots,\frac{n}{2}$.)

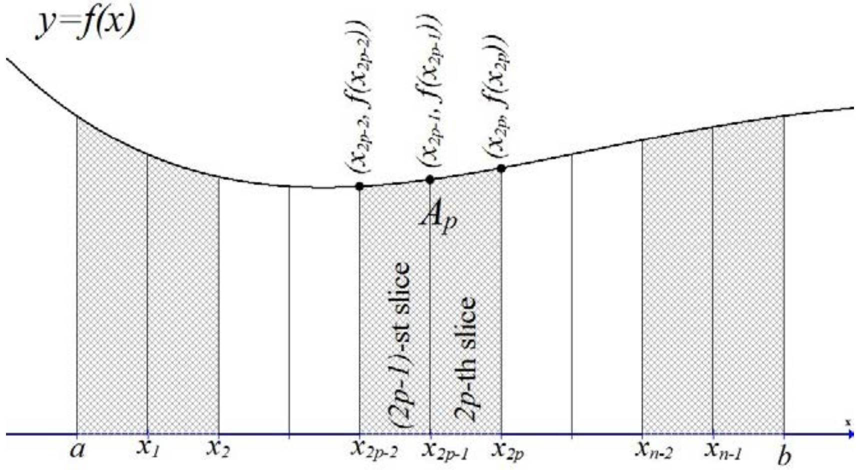


Areas Using Parabolas

Then, by approximating with parabolas,

$$A_p \approx \frac{h}{3}\left(y_{2p-2} + 4y_{2p-1} + y_{2p}\right)$$

where $y_k = f(x_k)$.



Areas Using Parabolas

Adding these approximations together, we get that the area, A, under the curve $y = f(x)$ on $[a, b]$ is approximately:

$$A = A_1 + A_2 + \cdots + A_{n/2}$$

$$\approx \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4) + \cdots + \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

$$= \frac{h}{3}[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \cdots + (y_{n-2} + 4y_{n-1} + y_n)]$$

$$= \frac{h}{3}[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n]$$

Simpson's Rule

To approximate $\int_a^b f(x)\,dx$, use

$$S = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n).$$

where

$$n \text{ is even}$$

$$\Delta x = \frac{b-a}{n}$$

$$x_k = a + k \cdot \Delta x \text{ (the partition points)}$$

and

$$y_k = f(x_k).$$

Example

Approximate

$$\int_1^2 \frac{1}{x}\,dx$$

by using Simpson's Rule with $n = 10$.

Example (continued)

Solution:

$$\int_{1}^{2} \frac{1}{x} dx$$

$$f(x) = \frac{1}{x}$$

$$[a, b] = [1,2]$$

$$n = 10$$

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{10} = \frac{1}{10}$$

$$x_k = a + k \cdot \Delta x = 1 + \frac{k}{10}$$

$$y_k = f(x_k) = \frac{1}{x_k}$$

$$= \frac{1}{1 + \frac{k}{10}} = \frac{10}{10 + k}$$

Example (continued)

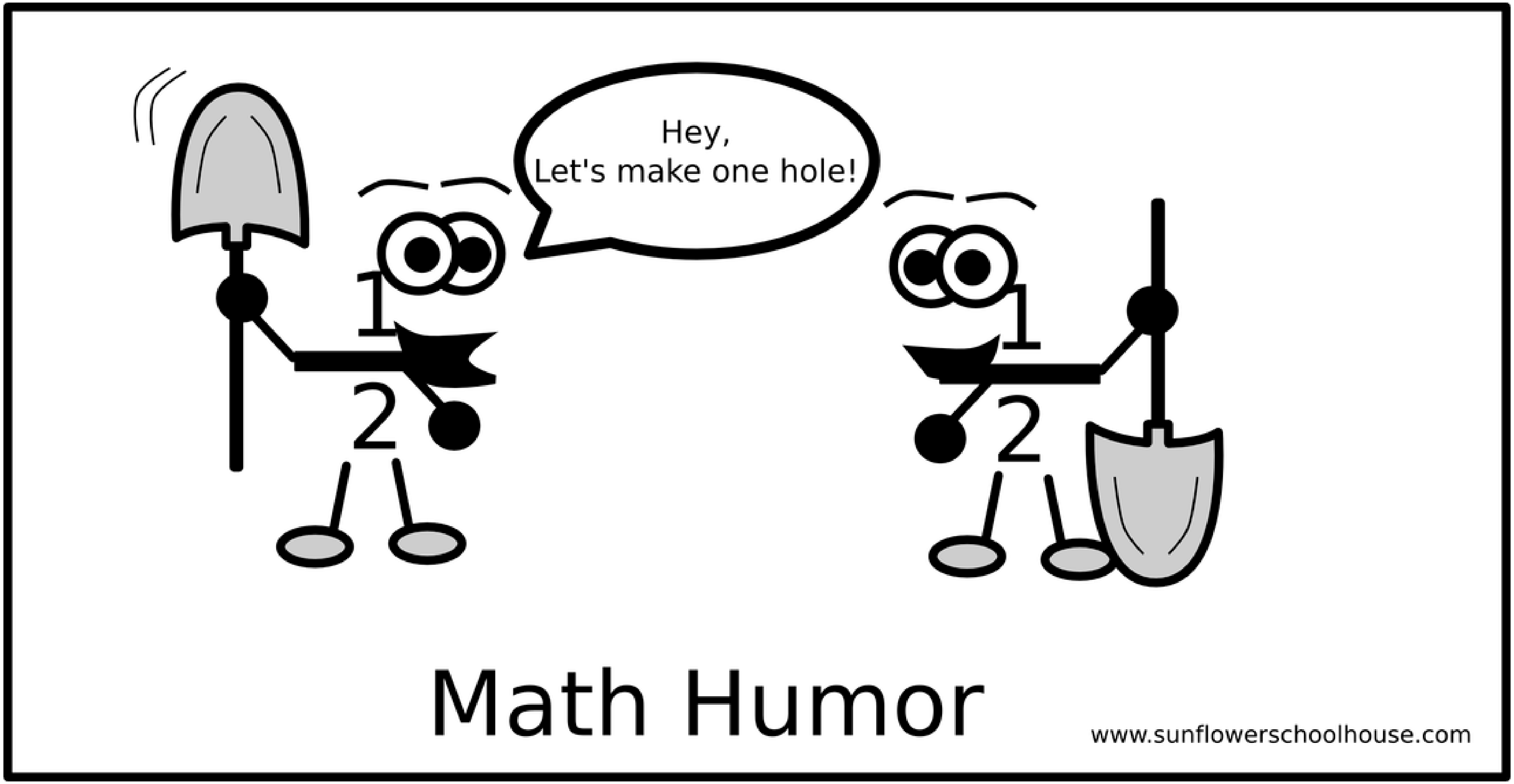
$$\int_1^2 \frac{1}{x}\,dx \approx \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n)$$

$$= \frac{1/10}{3}\left(\frac{10}{10+0} + 4\cdot\frac{10}{10+1} + 2\cdot\frac{10}{10+2} + 4\cdot\frac{10}{10+3} + 2\cdot\frac{10}{10+4} + 4\cdot\frac{10}{10+5} + 2\cdot\frac{10}{10+6} + 4\cdot\frac{10}{10+7} + 2\cdot\frac{10}{10+8} + 4\cdot\frac{10}{10+9} + \frac{10}{10+10}\right)$$

$$= \frac{1}{30}\left(1 + \frac{40}{11} + \frac{20}{12} + \frac{40}{13} + \frac{20}{14} + \frac{40}{15} + \frac{20}{16} + \frac{40}{17} + \frac{20}{18} + \frac{40}{19} + \frac{1}{2}\right)$$

$$\approx 0.69315$$

(Note: $\int_1^2 \frac{1}{x}\,dx = \ln 2 - \ln 1 = \ln 2 \approx .69315$ and the Trapezoid Rule gave us $\int_1^2 \frac{1}{x}\,dx \approx 0.69377$. So Simpson's Rule was more accurate than the Trapezoid Rule.)



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