Numerical Integration

Part 3:

Error Analysis

Error Analysis

Whenever we use an approximation technique, the issue arises as to how accurate the approximation might be.

The following theorem gives formulas for estimating the errors when using the Trapezoid Rule and Simpson's Rule.

The **error** is the difference between the approximation obtained by the rule and the actual value of the definite integral $\int_a^b f(x)\,dx$.

Error Estimates in the Trapezoid and Simpson's Rules

Let K be any number such that $|f''(x)| \leq K$ and let M be any number such that $\left|f^{(4)}(x)\right| \leq M$ for all x in $[a, b]$.

Then

$$|E_S| \leq \frac{M(b-a)^5}{180n^4}$$

and

$$|E_T| \leq \frac{K(b-a)^3}{12n^2}$$

where E_S is the error from Simpson's Rule and E_T is the error from the Trapezoid Rule.

Example

For the integral

$$\int_{1}^{2} \frac{1}{x} dx$$

a) Show that for $n = 10$, $|E_T| \leq \frac{2}{12\times10^2} \approx 0.00167$ and $|E_S| \leq \frac{24}{180\times10^4} \approx 0.0000133.$

b) How large should n be to ensure that $|E_S| \leq 0.0002$?

c) Answer (b) for the Trapezoid Rule.

Example (continued)

Solution (a):

Since $f(x) = \frac{1}{x} = x^{-1}$

$$f'(x) = -x^{-2} = -\frac{1}{x^2}$$
$$f''(x) = 2x^{-3}$$
$$f'''(x) = -6x^{-4}$$
$$f^{(4)}(x) = 24x^{-5} = \frac{24}{x^5}.$$

That means on $[1,2]$:

$$\frac{1}{4} \leq f''(x) \leq 2$$
$$\frac{3}{4} = \frac{24}{32} \leq f^{(4)}(x) \leq 24$$

Example (continued)

That is

$$|f''(x)| \leq 2 = K \text{ and } \left|f^{(4)}(x)\right| \leq 24 = M$$

So

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} = \frac{2(2-1)^3}{12(10)^2} = \frac{2}{12 \times 10^2}$$

and

$$|E_S| \leq \frac{M(b-a)^5}{180n^4} = \frac{24(2-1)^5}{180(10)^4} = \frac{24}{180 \times 10^4}$$

Example (continued)

Solution (b):

$$|E_S| \le \frac{M(b-a)^5}{180n^4} = \frac{24(2-1)^5}{180n^4} = \frac{24}{180n^4}$$

We want $|E_S| \le 0.0002$, so we just need $\frac{24}{180n^4} \le 0.0002$.

$$\frac{24}{180n^4} \le 0.0002$$

$$\frac{24}{180(0.0002)} \le n^4$$

$$5.08 \approx \sqrt[4]{\frac{24}{180(0.0002)}} \le n$$

$$\boxed{n = 6}$$

Example (continued)

<u>Solution (c)</u>:

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} = \frac{2(2-1)^3}{12n^2} = \frac{2}{12n^2}$$

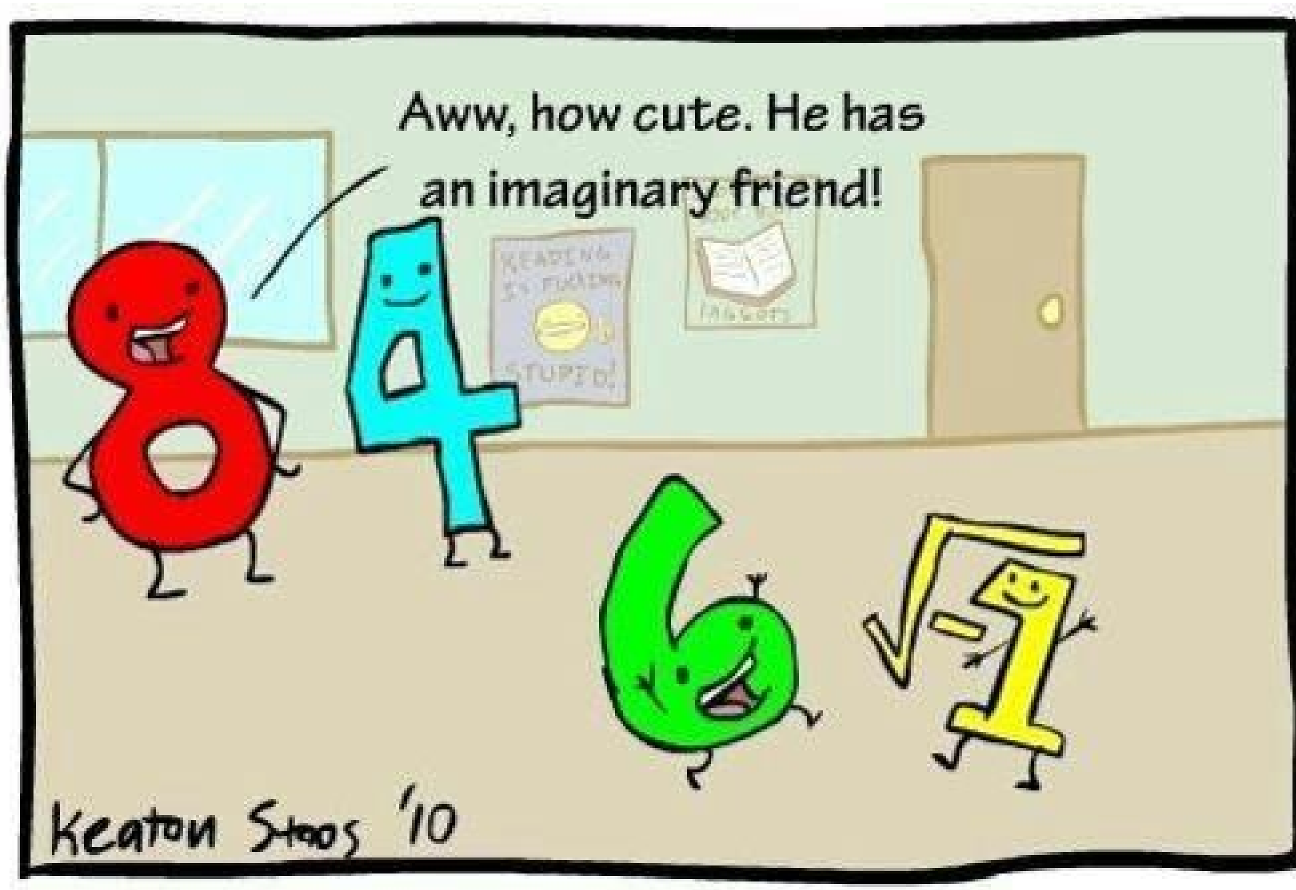
We want $|E_T| \leq 0.0002$, so we just need $\frac{2}{12n^2} \leq 0.0002$.

$$\frac{2}{12n^2} \leq 0.0002$$

$$\frac{2}{12(0.0002)} \leq n^2$$

$$28.8 \approx \sqrt{\frac{2}{12(0.0002)}} \leq n$$

$$\boxed{n = 29}$$



http://katycouponers.blogspot.com/2011/01/warning-way-off-topi-c-no-deal-posted.html