

Improper Integrals

Part 1: Introduction and Integrals with Infinite Limits of Integration

Introduction

Right now

$$\int_0^5 \frac{1}{(x-2)^{2/3}} dx$$

does not exist because the integrand is not bounded on $[0,5]$.

$$\left(\lim_{x \rightarrow 2} (x-2)^{-2/3} = \infty \right)$$

Improper Integrals

We will extend definite integrals to include:

- Integrals with infinite limits of integration
- Integrals which are not bounded on the interval

These are called **improper integrals**.

Integrals with Infinite Limits of Integration

Let f be continuous on $[a, +\infty)$. Then

$$\int_a^{+\infty} f(x) dx = \lim_{l \rightarrow +\infty} \int_a^l f(x) dx$$

- If this limit exists, the improper integral is said to **converge**, and the value of the limit is the value of the integral.
- If this limit does not exist, then the improper integral **diverges**, and the integral has no value.

Integrals with Infinite Limits of Integration (continued)

Let f be continuous on $(-\infty, b]$. Then

$$\int_{-\infty}^b f(x) dx = \lim_{l \rightarrow -\infty} \int_l^b f(x) dx$$

- If this limit exists, the improper integral is said to **converge**, and the value of the limit is the value of the integral.
- If this limit does not exist, then the improper integral **diverges**, and the integral has no value.

Integrals with Infinite Limits of Integration (continued)

Let f be continuous on $(-\infty, +\infty)$. Then

$$\int_{-\infty}^{+\infty} f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^{+\infty} f(x) dx$$

(Note: We often choose $c = 0$.)

- If both integrals on the right exist, then

$\int_{-\infty}^{+\infty} f(x) dx$ is said to **converge**.

- If either integral on the right does not exist, then

$\int_{-\infty}^{+\infty} f(x) dx$ **diverges**.

Example 1

Does the following improper integral converge or diverge? If it converges, evaluate the integral.

$$\int_1^{+\infty} \frac{1}{x} dx$$

Example 1 (continued)

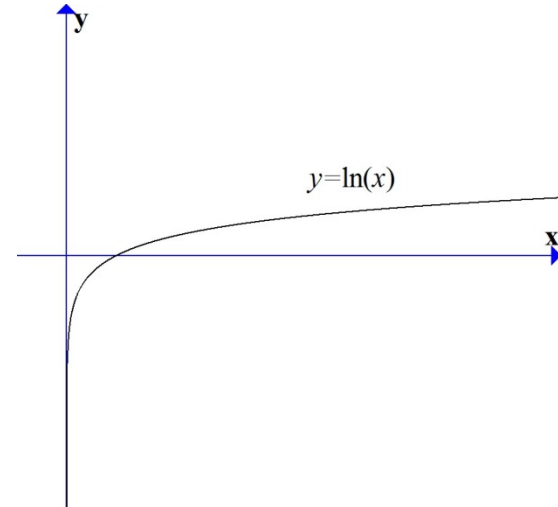
Solution:

$$\int_1^{+\infty} \frac{1}{x} dx = \lim_{l \rightarrow +\infty} \int_1^l \frac{1}{x} dx$$

$$= \lim_{l \rightarrow +\infty} \left[\ln|x| \Big|_1^l \right] = \lim_{l \rightarrow +\infty} [\ln|l| - \ln|1|]$$

$$= +\infty - 0 = +\infty$$

Therefore $\int_1^{+\infty} \frac{1}{x} dx$ diverges.



Example 2

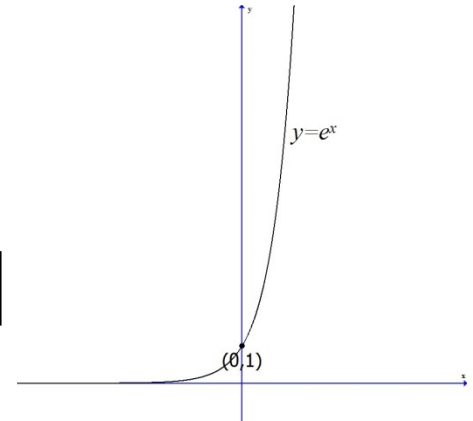
Does the following improper integral converge or diverge? If it converges, evaluate the integral.

$$\int_{-\infty}^0 e^x dx$$

Example 2 (continued)

Solution:

$$\begin{aligned}\int_{-\infty}^0 e^x dx &= \lim_{l \rightarrow -\infty} \int_l^0 e^x dx \\ &= \lim_{l \rightarrow -\infty} \left[e^x \Big|_l^0 \right] = \lim_{l \rightarrow -\infty} [e^0 - e^l] \\ &= 1 - 0 = 1\end{aligned}$$



Therefore $\int_{-\infty}^0 e^x dx$ converges and $\int_{-\infty}^0 e^x dx = 1$.

Example 3

Does the following improper integral converge or diverge? If it converges, evaluate the integral.

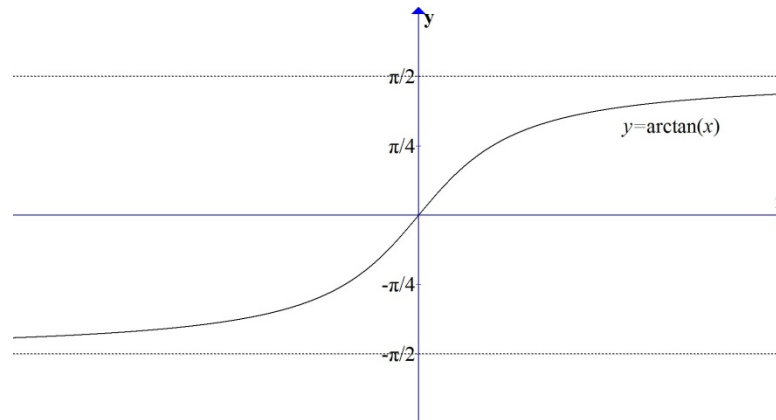
$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2} dx$$

Example 3 (continued)

Solution:

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{1}{1+x^2} dx &= \int_{-\infty}^0 \frac{1}{1+x^2} dx + \int_0^{+\infty} \frac{1}{1+x^2} dx \\ &= \lim_{l \rightarrow -\infty} \int_l^0 \frac{1}{1+x^2} dx + \lim_{l \rightarrow +\infty} \int_0^l \frac{1}{1+x^2} dx \\ &= \lim_{l \rightarrow -\infty} \left[\tan^{-1} x \Big|_l^0 \right] + \lim_{l \rightarrow +\infty} \left[\tan^{-1} x \Big|_0^l \right] \\ &= \lim_{l \rightarrow -\infty} \left[\tan^{-1} 0 - \tan^{-1} l \right] + \lim_{l \rightarrow +\infty} \left[\tan^{-1} l - \tan^{-1} 0 \right]\end{aligned}$$

Example 3 (continued)



$$\begin{aligned} \lim_{l \rightarrow -\infty} [\tan^{-1} 0 - \tan^{-1} l] + \lim_{l \rightarrow +\infty} [\tan^{-1} l - \tan^{-1} 0] \\ = \left[0 - \left(-\frac{\pi}{2} \right) \right] - \left[\frac{\pi}{2} - 0 \right] = \pi \end{aligned}$$

Therefore

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2} dx \text{ converges and } \int_{-\infty}^{+\infty} \frac{1}{1+x^2} dx = \pi.$$

After explaining to a student through various lessons and examples that:

$$\lim_{x \rightarrow 8} \frac{1}{x-8} = \infty$$

I tried to check if she really understood that, so I gave her a different example.

This was the result:

$$\lim_{x \rightarrow 5} \frac{1}{x-5} = \infty$$

<http://www.calculushumor.com/3/category/limits/2.html>