

Improper Integrals

Part 3:

Tests for Convergence and
Divergence

Useful Fact #1

First, if $n > 1$ and $a > 0$, then

$$\begin{aligned}\int_a^\infty \frac{1}{x^n} dx &= \lim_{l \rightarrow \infty} \int_a^l \frac{1}{x^n} dx \\ &= \lim_{l \rightarrow \infty} \left(\frac{1}{-n+1} x^{-n+1} \Big|_a^l \right) = \frac{1}{-n+1} \lim_{l \rightarrow \infty} \left(\frac{1}{x^{n-1}} \Big|_a^l \right) \\ &= \frac{1}{-n+1} \lim_{l \rightarrow \infty} \left(\frac{1}{l^{n-1}} - \frac{1}{a^{n-1}} \right) \\ &= \frac{1}{-n+1} \left(0 - \frac{1}{a^{n-1}} \right) = \frac{1}{a^{n-1}(n-1)}\end{aligned}$$

$$\int_a^\infty \frac{1}{x^n} dx \text{ converges for } n > 1 \text{ and } a > 0$$

Useful Fact #2

First, if $n < 1$ and $a > 0$, then

$$\begin{aligned}\int_a^\infty \frac{1}{x^n} dx &= \lim_{l \rightarrow \infty} \int_a^l \frac{1}{x^n} dx \\ &= \lim_{l \rightarrow \infty} \left(\frac{1}{1-n} x^{1-n} \Big|_a^l \right) \\ &= \frac{1}{1-n} \lim_{l \rightarrow \infty} (l^{1-n} - a^{1-n}) \\ &= \infty\end{aligned}$$

$\int_a^\infty \frac{1}{x^n} dx$ diverges for $n < 1$ and $a > 0$

Useful Fact #3

First, if $n = 1$ and $a > 0$, then

$$\int_a^{\infty} \frac{1}{x} dx = \lim_{l \rightarrow \infty} \int_a^l \frac{1}{x} dx$$

$$= \lim_{l \rightarrow \infty} \left(\ln x \Big|_a^l \right)$$

$$= \lim_{l \rightarrow \infty} (\ln l - \ln a) = \infty$$

$$\int_a^{\infty} \frac{1}{x} dx \text{ diverges for } a > 0$$

Direct Comparison Test

Let f and g be continuous on $[a, \infty)$ with $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then

1. $\int_a^\infty f(x) dx$ converges if $\int_a^\infty g(x) dx$ converges.
2. $\int_a^\infty g(x) dx$ diverges if $\int_a^\infty f(x) dx$ diverges.

Limit Comparison Test

If the positive functions f and g are continuous on $[a, \infty)$, and if

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = L, \quad 0 < L < \infty,$$

then

$$\int_a^{\infty} f(x) dx \text{ and } \int_a^{\infty} g(x) dx$$

both converge or both diverge.

Example 1

Perform a test to determine if

$$\int_3^{\infty} \frac{1}{x^3 + 7} dx$$

converges or diverges.

Solution:

We need to compare $\frac{1}{x^3 + 7}$ to a function $g(x)$ for which we know whether $\int_3^{\infty} g(x) dx$ converges or diverges.

Example 1 (continued)

If we let $g(x) = \frac{1}{x^3}$ then, by the Useful Fact #1, $\int_3^{\infty} g(x) dx$ converges.

Note that

$$0 \leq \frac{1}{x^3 + 7} \leq \frac{1}{x^3}.$$

So $\int_3^{\infty} \frac{1}{x^3 + 7} dx$ converges by the Direct Comparison Test with $\int_3^{\infty} \frac{1}{x^3} dx$.

Example 2

Perform a test to determine if

$$\int_2^{\infty} \frac{1}{\sqrt{x^2 - 1}} dx$$

converges or diverges.

Solution:

We need to compare $\frac{1}{\sqrt{x^2 - 1}}$ to a function $g(x)$ for which we know whether $\int_3^{\infty} g(x) dx$ converges or diverges.

Example 2 (continued)

If we let $g(x) = \frac{1}{x}$ then, by the Useful Fact #3, $\int_2^{\infty} g(x) dx$ diverges.

Since

$$g(x) = \frac{1}{x} < \frac{1}{\sqrt{x^2 - 1}}$$

We cannot use the Direct Comparison test which requires $0 \leq f(x) \leq g(x)$. Let's try the Limit Comparison Test.

Example 2 (continued)

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow \infty} \frac{\left(\frac{1}{\sqrt{x^2 - 1}} \right)}{\left(\frac{1}{x} \right)} \\ &= \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 - 1}}\end{aligned}$$

This is an indeterminate form of type $\frac{\infty}{\infty}$. We can use L'Hopital's Rule to evaluate this limit.

Another way is to re-write the limit as follows:

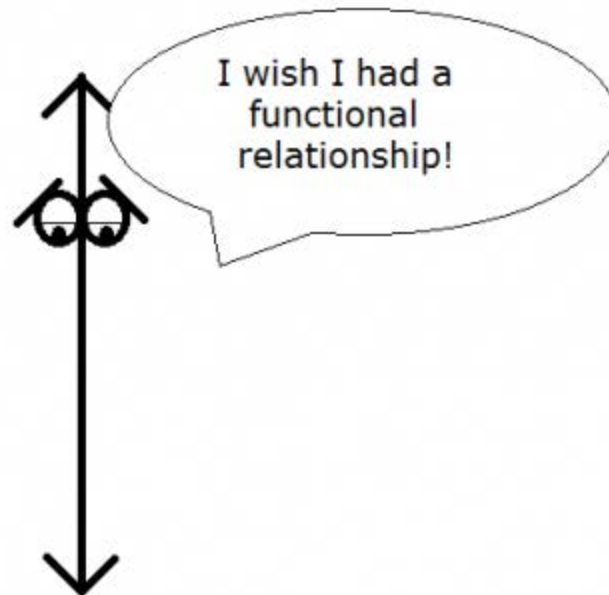
Example 2 (continued)

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 - 1}} &= \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 - 1}} \cdot \frac{1/x}{1/\sqrt{x^2}} \\ &= \lim_{x \rightarrow \infty} \frac{1}{\sqrt{1 - 1/x^2}} \\ &= \frac{1}{\sqrt{1 - 0}} = 1\end{aligned}$$

Example 2 (continued)

Since $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)}$ is positive and finite,

$\int_2^{\infty} \frac{1}{\sqrt{x^2-1}} dx$ diverges by the Limit Comparison
Test with $\int_2^{\infty} \frac{1}{x} dx$.



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