Sequences

Part 3

Recursive Sequence

A **recursive sequence** is a sequence where you are given a_1 and the value of a_{n+1} is determined by the values of $a_1, a_2, \ldots, a_n$.

Example 1

If we define the recursive sequence by $a_1 = 1$ and $a_{n+1} = a_n + 1$, then:

$$a_1 = 1$$
$$a_2 = a_1 + 1 = 1 + 1 = 2$$
$$a_3 = a_2 + 1 = 2 + 1 = 3$$
$$a_n = n$$
$$a_{n+1} = a_n + 1 = n + 1$$

That is, $\{a_n\}_{n=1}^{+\infty} = \{n\}_{n=1}^{+\infty}$

Example 2

If we define the recursive sequence by $a_1 = 1$ and $a_{n+1} = a_n(n+1)$, then:

$$a_1 = 1$$
$$a_2 = a_1 \cdot 2 = 1 \cdot 2$$
$$a_3 = a_2 \cdot 3 = 1 \cdot 2 \cdot 3$$
$$a_n = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n$$
$$a_{n+1} = a_n \cdot (n+1) = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n \cdot (n+1)$$

Notation: "n factorial" is

$$n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n$$

or equivalently

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1$$

$$a_1 = 1!$$
$$a_2 = 1! \cdot 2 = 2!$$
$$a_3 = 2! \cdot 3 = 3!$$
$$a_n = n!$$
$$a_{n+1} = n! \cdot (n+1) = (n+1)!$$

$$\{a_n\}_{n=1}^{+\infty} = \{n!\}_{n=1}^{+\infty}$$

Monotone Sequences

$\{a_n\}_{n=1}^{+\infty}$ is called:

- **Increasing** if $a_1 < a_2 < a_3 < \cdots < a_n < \cdots$
- **Nondecreasing** if $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq \cdots$
- **Decreasing** if $a_1 > a_2 > a_3 > \cdots > a_n > \cdots$
- **Nonincreasing** if $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots$

- If a sequence is nondecreasing or nonincreasing, then it is a **monotone** sequence.
- If a sequence is decreasing or increasing, then it is a **strictly monotone** sequence.

Theorem – Part 1

For a nondecreasing sequence $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_n \leq \cdots$ there are two possibilities:

1. There is a constant M such that $a_n \leq M$ for all n, in which case the sequence converges to a limit L satisfying $L \leq M$. The smallest such M is called the **least upper bound**.
2. No such constant exists, in which case $\lim_{n \to \infty} a_n = +\infty$.

Theorem – Part 2

For a nonincreasing sequence $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots$ there are two possibilities:

1. There is a constant M such that $a_n \geq M$ for all n, in which case the sequence converges to a limit L satisfying $L \geq M$. The largest such M is called the **greatest lower bound**.
2. No such constant exists, in which case $\lim_{n \to \infty} a_n = -\infty$.

Theorem Summary

If $\{a_n\}_{n=1}^{+\infty}$ is monotonic and bounded then it converges.

Example 3

Show $\left\{\frac{5^n}{n!}\right\}_{n=1}^{+\infty}$ converges.

Solution:
Since $a_n > 0$ for all n,

$$a_n \leq a_{n+1} \Leftrightarrow 1 \leq \frac{a_{n+1}}{a_n}$$

and

$$a_n \geq a_{n+1} \Leftrightarrow 1 \geq \frac{a_{n+1}}{a_n}$$

Example 3 (continued)

$$a_n = \frac{5^n}{n!} \qquad a_{n+1} = \frac{5^{n+1}}{(n+1)!}$$

Since

$$\frac{a_{n+1}}{a_n} = \frac{\left(\frac{5^{n+1}}{(n+1)!}\right)}{\left(\frac{5^n}{n!}\right)}$$

$$= \frac{5^{n+1}}{(n+1)!} \cdot \frac{n!}{5^n}$$

Example 3 (continued)

$$\frac{a_{n+1}}{a_n} = \frac{5^{n+1}}{(n+1)!} \cdot \frac{n!}{5^n} = \frac{5^n \cdot 5}{n!\,(n+1)} \cdot \frac{n!}{5^n}$$

$$= \frac{5}{n+1}$$

For $n = 1, 2, 3$ we have

$$1 < \frac{a_{n+1}}{a_n}$$

For $n \geq 4$ we have

$$1 \geq \frac{a_{n+1}}{a_n}$$

Example 3 (continued)

So, if we discard the first 3 terms of the sequence (which won't affect convergence) the resulting sequence is decreasing.

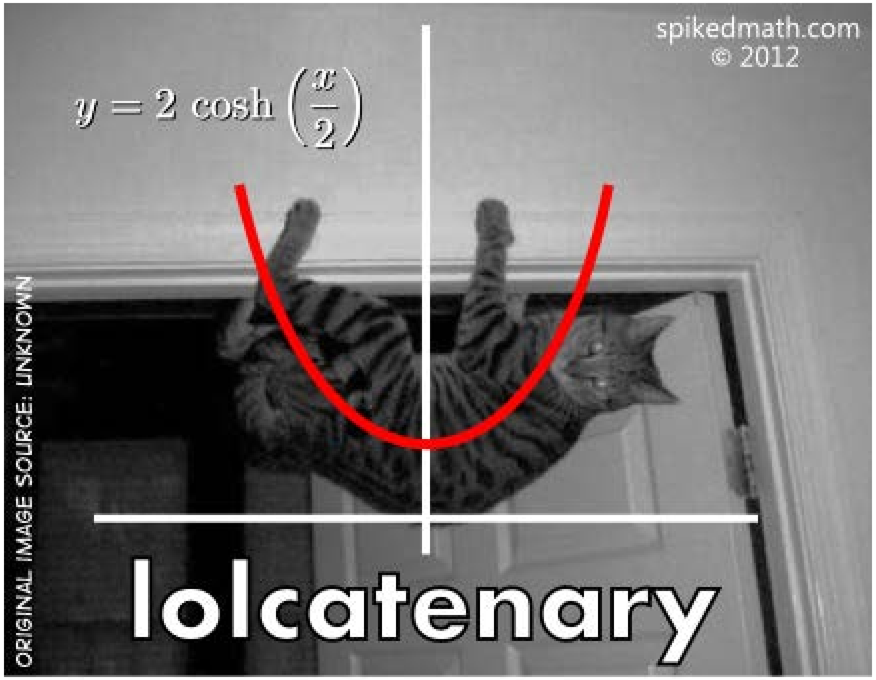
Since all of the terms are positive,

the sequence is bounded below by 0.

Therefore, the sequence converges.

Background: A **lolcat** is an image combining a photograph of a cat with text intended to contribute humor. The text is often idiosyncratic and grammatically incorrect, and its use in this way is known as "lolspeak" or "kitty pidgin".

http://spikedmath.com/503.html



In physics and geometry, the lolcatenary is the curve that an idealized hanging lolcat assumes under its own weight when supported only at its ends.