The Binomial Series

The Binomial Series

If m is a real number then the Maclaurin series for $(1+x)^m$ is called the **binomial series**.

$$1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots + \frac{m(m-1)(m-2)\cdots\big(m-(k-1)\big)}{3!}x^k + \cdots$$

This converges to $(1+x)^m$ for $|x| < 1$.

Notation

m is any real number, k is any non-negative integer

$$\binom{m}{0} = 1$$

$$\binom{m}{1} = m$$

$$\binom{m}{k} = \frac{m(m-1)(m-2)\cdots\big(m-(k-1)\big)}{k!}, k \geq 3$$

The Binomial Series

If m is a real number then

$$(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k$$

for $|x| < 1$.

Example 1

Express

$$\frac{1}{\sqrt{1+x}} = (1+x)^{-1/2}$$

as a binomial series.

<u>Solution</u>:

$$m = -\frac{1}{2}$$

$$(1+x)^{-1/2} = \sum_{k=0}^{\infty} \binom{-1/2}{k} x^k$$

Example 1 (continued)

$$\binom{-1/2}{0} = 1$$

$$\binom{-1/2}{1} = -\frac{1}{2}$$

$$\binom{-1/2}{2} = \frac{\left(-\frac{1}{2}\right)\left(\left(-\frac{1}{2}\right)-1\right)}{2!} = \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!} = (-1)^2\,\frac{1\cdot 3}{2^2\cdot 2!}$$

$$\binom{-1/2}{3} = \frac{\left(-\frac{1}{2}\right)\left(\left(-\frac{1}{2}\right)-1\right)\left(\left(-\frac{1}{2}\right)-2\right)}{3!} = \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!} = (-1)^3\,\frac{1\cdot 3\cdot 5}{2^3\cdot 3!}$$

Example 1 (continued)

$$\binom{-1/2}{4} = \frac{\left(-\frac{1}{2}\right)\left(\left(-\frac{1}{2}\right)-1\right)\left(\left(-\frac{1}{2}\right)-2\right)\left(\left(-\frac{1}{2}\right)-3\right)}{4!} = \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{4!} = (-1)^4\frac{1\cdot 3\cdot 5\cdot 7}{2^4\cdot 4!}$$

⋮

$$\binom{-1/2}{k} = \frac{\left(-\frac{1}{2}\right)\left(\left(-\frac{1}{2}\right)-1\right)\left(\left(-\frac{1}{2}\right)-2\right)\cdots\left(\left(-\frac{1}{2}\right)-(k-1)\right)}{k!} = \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)\cdots\left(\left(-\frac{1}{2}\right)-\frac{2k-2}{2}\right)}{k!}$$

$$= (-1)^k\frac{1\cdot 3\cdot 5\cdot 7\cdots(2k-1)}{2^k\cdot k!}$$

Example 1 (continued)

Putting this together we get

$$(1+x)^{-1/2} = \sum_{k=0}^{\infty} \binom{-1/2}{k} x^k$$

$$= 1 - \frac{1}{2}x + \frac{1\cdot 3}{2^2\cdot 2!}x^2 - \frac{1\cdot 3\cdot 5}{2^3\cdot 3!}x^3 + \frac{1\cdot 3\cdot 5\cdot 7}{2^4\cdot 4!}x^4$$

$$+ \cdots + (-1)^k \frac{1\cdot 3\cdot 5\cdot 7\cdots(2k-1)}{2^k\cdot k!}x^k + \cdots$$

EVERY TIME YOU DO THIS:

$$f(x) = \frac{\cancel{x^2} + 2x + 1}{\cancel{x^2} + 3}$$

$$= \frac{2x+1}{3}$$

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