# Infinite Series

Part 1

# Infinite Series

An **infinite series** is an expression of the form

$$u_1 + u_2 + u_3 + \cdots + u_k + \cdots$$

$$= \sum_{k=1}^{\infty} u_k$$

# $n$-th Partial Sums

We will define the ***n*-th partial sum** of the series to be

$$s_n = \sum_{k=1}^{n} u_k$$

The sequence $\{s_n\}_{n=1}^{\infty}$ is called the **sequence of partial sums**.

# Sum of a Series

If $\{s_n\}_{n=1}^{\infty}$ converges to a limit $S$, then the series **converges** and $S$ is the **sum** of the series.

$$S = \lim_{n\to\infty} s_n = \sum_{k=1}^{\infty} u_k$$

Otherwise, the series **diverges** and has no sum.

# Example 1

The infinite series

$$\frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^k} + \cdots$$

$$= \sum_{k=1}^{\infty} \frac{3}{10^k}$$

has the following partial sums:

# Example 1 (continued)

$$s_1 = \frac{3}{10}$$

$$s_2 = \frac{3}{10} + \frac{3}{10^2} = \frac{33}{100}$$

$$s_3 = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} = \frac{333}{1000}$$

$$\vdots$$

$$s_n = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^n}$$

# Example 1 (continued)

Here is a neat trick to remember to get a formula for $s_n$:

$$s_n = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^n}$$

$$\frac{1}{10} s_n = \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \cdots + \frac{3}{10^{n+1}}$$

Subtracting we get:

$$s_n - \frac{1}{10} s_n = \frac{3}{10} - \frac{3}{10^{n+1}}$$

# Example 1 (continued)

$$s_n - \frac{1}{10}s_n = \frac{3}{10} - \frac{3}{10^{n+1}}$$

$$\frac{9}{10}s_n = \frac{3}{10}\left(1 - \frac{1}{10^n}\right)$$

$$s_n = \frac{1}{3}\left(1 - \frac{1}{10^n}\right)$$

# Example 1 (continued)

Now we can see

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1}{3}\left(1 - \frac{1}{10^n}\right) = \frac{1}{3}$$

That is:

$$\frac{1}{3} = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots + \frac{3}{10^k} + \cdots$$

# Example 2

Does the series

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

converge or diverge? What is its sum?

# Example 2 (continued)

Solution:

$$s_1 = 1$$
$$s_2 = 1 - 1 = 0$$
$$s_3 = 1 - 1 + 1 = 1$$
$$s_4 = 1 - 1 + 1 - 1 = 0$$

etc.

So,

$$s_1, s_2, s_3, s_4, \ldots = 1, 0, 1, 0, \ldots$$

Since this sequence diverges, the series diverges and has no sum.

# Example 3

Does the series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots$$

converge or diverge? If it converges, what is its sum?

# Example 3 (continued)

Solution:

$$s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)}$$

$$= \sum_{k=1}^{n} \left( \frac{1}{k} - \frac{1}{k+1} \right)$$

$$= \left( \frac{1}{1} - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \cdots + \left( \frac{1}{n} - \frac{1}{n+1} \right)$$

$$= 1 - \frac{1}{n+1}$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left( 1 - \frac{1}{n+1} \right) = 1$$

The series converges and $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1$.


http://math.sfsu.edu/beck/images/foxtrot.math.hw.how.to.get.answers.gif