Integral Test

Part 1

Theorem 1

Suppose

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \cdots + u_k + \cdots$$

has positive terms.

(Then $s_1 < s_2 < s_3 < \cdots < s_k < \cdots$.)

If there exists an M such that

$$s_n = u_1 + u_2 + u_3 + \cdots + u_n \leq M$$

for every n,

then the series converges

and the sum S satisfies $S \leq M$.

Otherwise, the series diverges.

The Integral Test

Let $\sum u_k$ be a series with positive terms, and let $f(x)$ be the function that results when k is replaced by x in the formula for u_k.

If f is decreasing and continuous for $x \geq N$, then

$$\sum_{k=N}^{\infty} u_k \text{ and } \int_N^{\infty} f(x)\,dx$$

both converge or both diverge.

Example 1

Determine whether

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

converges or diverges.

<u>Solution</u>:

$$u_k = \frac{1}{k^2} \text{ and } f(x) = \frac{1}{x^2}$$

$f(x)$ is obviously decreasing and continuous for $x \geq 1$.

Example 1 (continued)

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x^2}\,dx$$

$$= \lim_{l\to\infty} \int_1^l \frac{1}{x^2}\,dx$$

$$= \lim_{l\to\infty} \left.\frac{-1}{x}\right|_1^l$$

$$= \lim_{l\to\infty} \left(\frac{-1}{l} - \frac{-1}{1}\right)$$

$$= 1$$

Since the integral converges, the series converges.

Note:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \neq \int_1^\infty \frac{1}{x^2}\,dx = 1$$

Example 2

Determine whether the series

$$\sum_{k=1}^{\infty} \frac{k}{e^{k^2}} = \frac{1}{e} + \frac{2}{e^4} + \frac{3}{e^9} + \cdots + \frac{k}{e^{k^2}} + \cdots$$

converges or diverges.

<u>Solution</u>:

$$u_k = \frac{k}{e^{k^2}} \text{ and } f(x) = \frac{x}{e^{x^2}} = xe^{-x^2}$$

$f(x)$ is obviously continuous, but is it decreasing?

Example 2 (continued)

Since

$$f'(x) = \frac{d}{dx}\left(xe^{-x^2}\right)$$
$$= xe^{-x^2}(-2x) + e^{-x^2}$$
$$= (-2x^2 + 1)e^{-x^2}$$

and

$$(-2x^2 + 1)e^{-x^2} = \frac{1 - 2x^2}{e^{x^2}} < 0 \text{ for } x \geq 1$$

we know that f is decreasing for $x \geq 1$.

Example 2 (continued)

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{x}{e^{x^2}}\,dx$$
$$= \lim_{l\to\infty} \int_1^l \frac{x}{e^{x^2}}\,dx = \lim_{l\to\infty} \int_1^l xe^{-x^2}\,dx$$
$$= \lim_{l\to\infty} \left(-\frac{1}{2}\cdot e^{-x^2}\right)\Big|_1^l$$
$$= \lim_{l\to\infty} \left(\frac{-1}{2e^{l^2}} - \frac{-1}{2e^{1^2}}\right)$$
$$= \frac{1}{2e}$$

Since the integral converges, the series converges.

p-series

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{k^p} + \cdots$$

$$p > 0$$

Convergence of p-series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

- Converges if $p > 1$
- Diverges if $0 < p \leq 1$

Convergence of p-series (continued)

Proof:

We saw in the Improper Integrals videos that:

- $\int_a^\infty \frac{1}{x^n}\,dx$ converges for $n > 1$ and $a > 0$
- $\int_a^\infty \frac{1}{x^n}\,dx$ diverges for $n < 1$ and $a > 0$
- $\int_a^\infty \frac{1}{x}\,dx$ diverges for $a > 0$

Convergence of p-series (continued)

So, by the Integral Test:

- $\sum_{k=1}^{\infty} \frac{1}{k^p}, p > 1$ converges since $\int_1^{\infty} \frac{1}{x^p} dx$ converges
- $\sum_{k=1}^{\infty} \frac{1}{k^p}, p < 1$ diverges since $\int_1^{\infty} \frac{1}{x^p} dx$ diverges
- $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges since $\int_a^{\infty} \frac{1}{x} dx$ diverges

thus proving the Convergence of p-series.

Example 3

$$1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \cdots + \frac{1}{\sqrt[3]{k}} + \cdots$$

Diverges since it is a p-series with $p = \frac{1}{3} < 1$.

A couple of days ago,
when my math teacher asked,
"Any questions?"

I asked, "What is the meaning of life?"
She simply replied,
"The meaning of life is math."

Today, we realized that, in the alphabet
M is the 13th letter
A is the 1st letter
T is the 20th letter
And H is the 8th letter

$$13 + 1 + 20 + 8 = 42$$

http://24.media.tumblr.com/7bf98ef932958f6559effb5f7ed39549/tumblr_mjq72jBk2z1r0n60lo1_500.jpg