The Comparison Test

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Let $\sum a_k$ and $\sum b_k$ be series with positive terms and suppose

$$a_N \le b_N, a_{N+1} \le b_{N+1}, a_{N+2} \le b_{N+2}, \cdots,$$

a) If the "bigger series" $\sum b_k$ converges, then the "smaller series" $\sum a_k$ also converges.

b) On the other hand, if the "smaller series" $\sum a_k$ diverges, then the "bigger series" $\sum b_k$ also diverges.

Informal Principle #1

Suppose $\sum u_k$ is series with positive terms.

Constant terms in the denominator of u_k can usually be deleted without affecting the convergence of divergence of the series.

Example 1

<u>Guess</u> if the series converge or diverge.

$a)\ \sum_{k=1}^{\infty} \frac{1}{2^k+1}$

$b)\ \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}-2}$

$c)\ \sum_{k=1}^{\infty} \frac{1}{\left(k+\frac{1}{2}\right)^3}$

Example 1 (continued)

Solution:

(a) $\sum_{k=1}^{\infty} \frac{1}{2^k+1}$ we expect to behave like $\sum_{k=1}^{\infty} \frac{1}{2^k}$, which is a **convergent** geometric series $\left(a = \frac{1}{2}, r = \frac{1}{2}\right)$

Example 1 (continued)

(b) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}-2}$ we expect to behave like

$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$, which is a **divergent** p-series ($p = \frac{1}{2}$)

(c) $\sum_{k=1}^{\infty} \frac{1}{\left(k+\frac{1}{2}\right)^{3}}$ we expect to behave like

$\sum_{k=1}^{\infty} \frac{1}{k^{3}}$, which is a **convergent** p-series ($p = 3$)

Informal Principle #2

If a polynomial in k appears as a factor in the numerator or denominator of u_k,

all but the highest power of k in the polynomial may usually be deleted without affecting the convergence of divergence behavior of the series.

Example 2

<u>Guess</u> if the series converge or diverge.

$a)\ \sum_{k=1}^{\infty} \frac{1}{\sqrt{k^3+2k}}$

$b)\ \sum_{k=1}^{\infty} \frac{6k^4-2k^3+1}{k^5+k^2-2k}$

Example 2 (continued)

Solution:

(a) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^3+2k}}$ we expect to behave like

$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^3}}$, which is a **convergent** p-series ($p = \frac{3}{2}$).

(b) $\sum_{k=1}^{\infty} \frac{6k^4-2k^3+1}{k^5+k^2-2k}$ we expect to behave like

$\sum_{k=1}^{\infty} \frac{6k^4}{k^5} = 6\sum_{k=1}^{\infty} \frac{1}{k}$, which is a constant times the **divergent** harmonic series.

Example 3

Use the Comparison Test to determine whether

$$\sum_{k=1}^{\infty} \frac{1}{2k^2 + k}$$

converges or diverges.

<u>Solution</u>:

We expect this series to behave like $\sum_{k=1}^{\infty} \frac{1}{2k^2} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k^2}$, which is a constant times a convergent p-series $(p = 2)$.

Example 3 (continued)

Since we expect the series to converge, we want to find $\sum b_k$ such that $\sum b_k$ converges and

$$\frac{1}{2k^2 + k} \leq b_k.$$

Notice

$$\frac{1}{2k^2 + k} \leq \frac{1}{2k^2} \text{ for } k \geq 1.$$

Since $\sum_{k=1}^{\infty} \frac{1}{2k^2}$ converges, so does $\sum_{k=1}^{\infty} \frac{1}{2k^2+k}$ by the Comparison Test.

Example 4

Use the Comparison Test to determine whether

$$\sum_{k=1}^{\infty} \frac{1}{2k^2 - k}$$

converges or diverges.

<u>Solution</u>:

We expect this series to behave like $\sum_{k=1}^{\infty} \frac{1}{2k^2} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k^2}$, which is a constant times a convergent p-series $(p = 2)$.

Example 4 (continued)

Since we expect the series to converge, we want to find $\sum b_k$ such that $\sum b_k$ converges and

$$\frac{1}{2k^2 - k} \leq b_k.$$

Unfortunately

$$\frac{1}{2k^2 - k} > \frac{1}{2k^2} \text{ for } k \geq 1.$$

So $\sum_{k=1}^{\infty} \frac{1}{2k^2}$ cannot be our choice for $\sum b_k$.

Example 4 (continued)

We want to decrease the denominator of $\frac{1}{2k^2-k}$.

If we try $b_k = \frac{1}{2k^2-k^2} = \frac{1}{k^2}$ we get

$$\frac{1}{2k^2 - k} \leq \frac{1}{2k^2 - k^2} = \frac{1}{k^2} \text{ for } k \geq 1.$$

Since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p-series ($p = 2$), our series $\sum_{k=1}^{\infty} \frac{1}{2k^2-k}$ converges by the Comparison Test.

Example 5

Use the Comparison Test to determine whether

$$\sum_{k=1}^{\infty} \frac{1}{k - \frac{1}{4}}$$

converges or diverges.

<u>Solution</u>:

We expect this series to behave like $\sum_{k=1}^{\infty} \frac{1}{k}$, which is the divergent harmonic series.

Example 5 (continued)

Since we expect the series to diverge, we want to find $\sum a_k$ such that $\sum a_k$ diverges and

$$a_k \leq \frac{1}{k - \frac{1}{4}}.$$

Notice

$$\frac{1}{k} \leq \frac{1}{k - \frac{1}{4}} \text{ for } k \geq 1.$$

Since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, our series $\sum_{k=1}^{\infty} \frac{1}{k - \frac{1}{4}}$ diverges by the Comparison Test.

Example 6

Use the Comparison Test to determine whether

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}+5}$$

converges or diverges.

<u>Solution</u>:

We expect this series to behave like $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$, which is a divergent p-series ($p = \frac{1}{2}$).

Example 6 (continued)

Since we expect the series to diverge, we want to find $\sum a_k$ such that $\sum a_k$ diverges and

$$a_k \leq \frac{1}{\sqrt{k}+5}.$$

Unfortunately,

$$\frac{1}{\sqrt{k}} \geq \frac{1}{\sqrt{k}+5} \text{ for } k \geq 1.$$

So $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ cannot be our choice for $\sum a_k$.

Example 6 (continued)

We want to increase the denominator of $\frac{1}{\sqrt{k}+5}$.

If we try $a_k = \frac{1}{\sqrt{k}+\sqrt{k}} = \frac{1}{2\sqrt{k}}$ we get

$$\frac{1}{\sqrt{k}+\sqrt{k}} = \frac{1}{2\sqrt{k}} \leq \frac{1}{\sqrt{k}+5} \text{ for } k \geq 25.$$

Since $\sum_{k=1}^{\infty} \frac{1}{2\sqrt{k}} = \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ is a constant times a divergent p-series ($p = \frac{1}{2}$),

our series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}+5}$ diverges by the Comparison Test.

The Limit Comparison Test

Let $\sum a_k$ and $\sum b_k$ be series with positive terms and suppose

$$\rho = \lim_{k \to \infty} \frac{a_k}{b_k}$$

a) If ρ is finite and $\rho > 0$, then the series both converge or both diverge.

b) If $\rho = 0$ and $\sum b_k$ converges, then $\sum a_k$ converges.

c) If $\rho = \infty$ and $\sum b_k$ diverges, then $\sum a_k$ diverges.

Example 7

Use the Limit Comparison Test to determine whether

$$\sum_{k=1}^{\infty} \frac{3k^3 - 2k^2 + 4}{k^5 - k^3 + 2}$$

converges or diverges.

<u>Solution</u>:

We expect this series to behave like $\sum_{k=1}^{\infty} \frac{3k^3}{k^5} = \sum_{k=1}^{\infty} \frac{3}{k^2} = 3 \sum_{k=1}^{\infty} \frac{1}{k^2}$, which is a constant times a convergent p-series $(p = 2)$.

Example 7 (continued)

$$\rho = \lim_{k\to\infty} \frac{\left(\frac{3k^3 - 2k^2 + 4}{k^5 - k^3 + 2}\right)}{\left(\frac{3}{k^2}\right)}$$

$$= \lim_{k\to\infty} \frac{3k^3 - 2k^2 + 4}{k^5 - k^3 + 2} \cdot \frac{k^2}{3}$$

Since this is a rational function with the degree of the numerator equal to the degree of the denominator (=5), this limit is equal to the ratio of the leading coefficients.

Example 7 (continued)

$$\rho = \lim_{k\to\infty} \frac{3k^3 - 2k^2 + 4}{k^5 - k^3 + 2} \cdot \frac{k^2}{3} = \frac{3}{3} = 1 > 0$$

So, by the Limit Comparison Test,

$\sum_{k=1}^{\infty} \frac{3k^3-2k^2+4}{k^5-k^3+2}$ converges.

Thomas Simpson (1720 - 1761)

Simpson was a successful text writer and did most of his work in probability. He taught at the Royal Military Academy in Woolwich. His first articles were published in the *Ladies' Diary*. Later he became editor of this popular journal.

Simpson's rule to approximate definite integrals was developed and used before he was born. It is another of history's beautiful quirks that one of the ablest mathematicians of the 18th century is remembered not for his own work or his textbooks but for a rule that was never his, that he never claimed, and that bears his name only because he happened to mention it in one of his books.