Alternating Series, Absolute and Conditional Convergence

Part 1

Alternating Series

Series whose terms are alternating positive and negative are called **alternating**.

$$\sum_{k=1}^{\infty}(-1)^k a_k \text{ or } \sum_{k=1}^{\infty}(-1)^{k+1} a_k \qquad a_k > 0$$

Alternating Series Test

An alternating series

$$\sum_{k=1}^{\infty}(-1)^k a_k \text{ or } \sum_{k=1}^{\infty}(-1)^{k+1} a_k \qquad a_k > 0$$

converges if the following two conditions are satisfied:

$a)$ $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_k \geq \cdots$

$b)$ $\lim_{k \to \infty} a_k = 0$

Alternating Harmonic Series

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{k+1} \frac{1}{k} + \cdots$$

Since $a_k = \frac{1}{k} > \frac{1}{k+1} = a_{k+1} > 0$

and $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{k} = 0$

the Alternating Harmonic Series converges.

Example 1

Does the alternating series

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{k+3}{k(k+1)}$$

converge or diverge?

<u>Solution</u>:

$$a_k = \frac{k+3}{k(k+1)}$$

Example 1 (continued)

$$a_k = \frac{k+3}{k(k+1)}$$

$$a_{k+1} = \frac{(k+1)+3}{(k+1)\big((k+1)+1\big)}$$

$$= \frac{k+4}{(k+1)(k+2)}$$

We want to show that $a_k \geq a_{k+1}$.

If $a_k \geq a_{k+1}$ then $1 \geq \frac{a_{k+1}}{a_k}$.

Example 1 (continued)

$$\frac{a_{k+1}}{a_k} = \frac{\left(\frac{k+4}{(k+1)(k+2)}\right)}{\left(\frac{k+3}{k(k+1)}\right)}$$

$$= \frac{(k+4)k(k+1)}{(k+1)(k+2)(k+3)}$$

$$= \frac{k(k+4)}{(k+2)(k+3)}$$

$$= \frac{k^2+4k}{k^2+5k+6}$$

$$= \frac{k^2+4k}{k^2+5k+6}$$

$$= \frac{k^2+4k}{(k^2+4k)+k+6} < 1$$

Example 1 (continued)

Therefore $1 \geq \frac{a_{k+1}}{a_k}$ and $a_k \geq a_{k+1}$.

Also, $\lim_{k \to \infty} a_k = \lim_{k \to \infty} \frac{k+3}{k(k+1)} = 0.$

So by the Alternating Series Test, the series converges.

The Alternating Series Estimation Theorem

If an alternating series satisfies the hypotheses of the alternating series test, and

if the sum S of the series is approximated by the nth partial sum s_n,

then the absolute value of the error is less than a_{n+1} (the first unused term).

Furthermore, the sum S lies between any two successive partial sums s_n and s_{n+1}, and

the remainder, $S - s_n$, has the same sign as the first unused term.

The Alternating Series Estimation Theorem (Summary)

$$S = \sum_{k=1}^{\infty} (-1)^k a_k \approx s_n = \sum_{k=1}^{n} (-1)^k a_k$$

$$\underbrace{|S - s_n|}_{\text{the error}} < \underbrace{a_{n+1}}_{\text{the first unused term}}$$

$$s_n \leq S \leq s_{n+1}$$

- $S - s_n$ is positive if a_{n+1} is positive
- $S - s_n$ is negative if a_{n+1} is negative

Example 2

How many terms should be used to estimate

$$\sum_{k=1}^{\infty} (-1)^k \frac{1}{k^2+3}$$

with an error of less than 0.001?

<u>Solution</u>:

Find the n such that $a_{n+1} < 0.001$.

Example 2 (continued)

$$a_{n+1} = \frac{1}{(n+1)^2 + 3} < 0.001 = \frac{1}{1000}$$

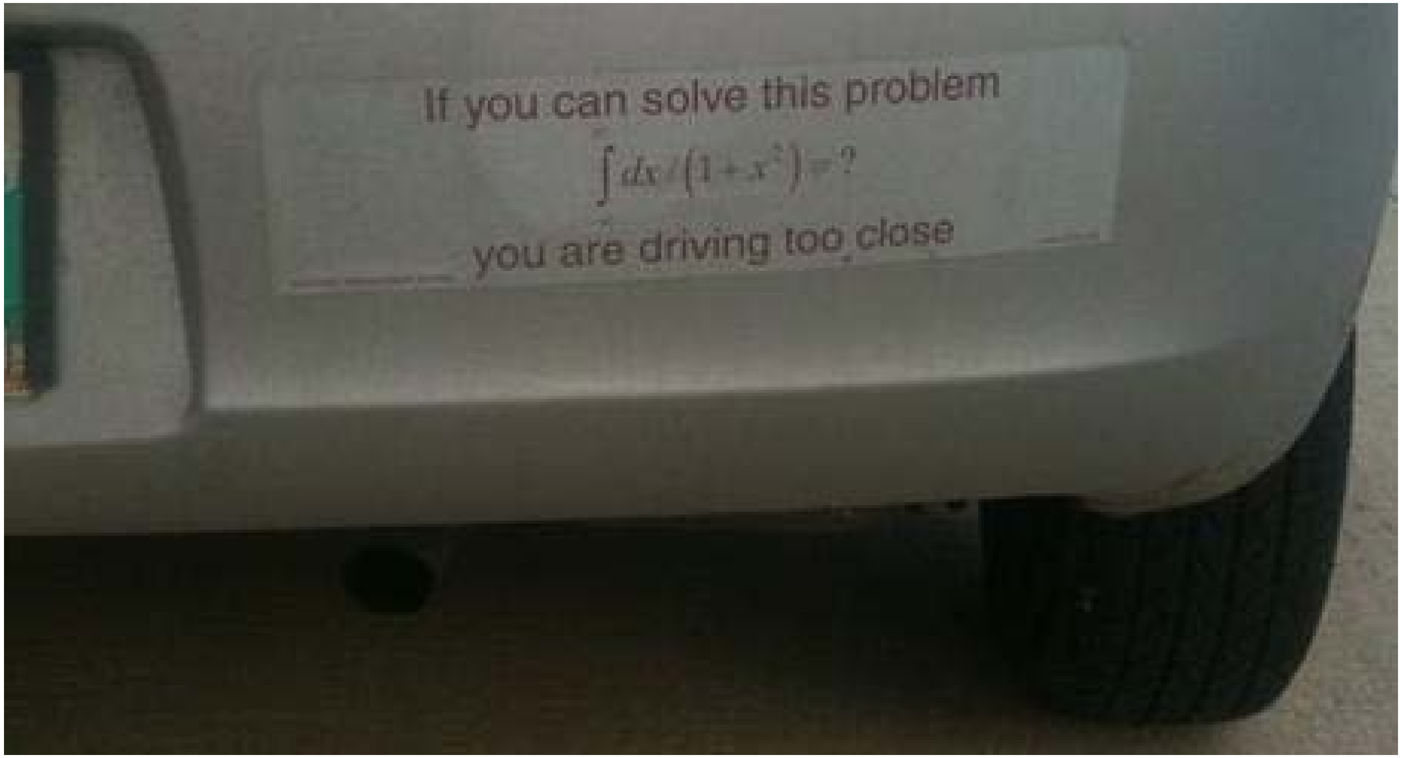
$$(n+1)^2 + 3 > 1000$$

$$(n+1)^2 > 997$$

$$n + 1 > \sqrt{997}$$

$$n > \sqrt{997} - 1 \approx 30.575$$

So 31 or more terms should be used.



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