Taylor and Maclaurin Series

Goal: Approximate $f(x)$

Suppose we want to approximate $f(x)$ by a polynomial on an interval centered at 0:

$$f(x) \approx p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$$

We will insist that:

$$\begin{gathered} f(0) = p(0) \\ f'(0) = p'(0) \\ f''(0) = p''(0) \\ \vdots \\ f^{(n)}(0) = p^{(n)}(0) \end{gathered}$$

Goal: Approximate $f(x)$

Since

$$p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$$
$$p'(x) = c_1 + 2c_2 x + \cdots + n c_n x^{n-1}$$
$$p''(x) = 2c_2 + \cdots + n(n-1) c_n x^{n-2}$$
$$\vdots$$
$$p^{(n)}(x) = n \cdot (n-1) \cdot \cdots \cdot 1 \cdot c_n = n!\, c_n$$

we get

$$f(0) = p(0) = c_0$$
$$f'(0) = p'(0) = c_1$$
$$f''(0) = p''(0) = 2c_2$$
$$\vdots$$
$$f^{(n)}(0) = p^{(n)}(0) = n!\, c_n$$

Goal: Approximate $f(x)$

Solving for c_k's we get:

$$f(0) = c_0$$

$$f'(0) = c_1$$

$$\frac{f''(0)}{2} = c_2$$

$$\vdots$$

$$\frac{f^{(n)}(0)}{n!} = c_n$$

n-th Maclaurin Polynomial

If f can be differentiated n times at $x = 0$, then we define the **n-th Maclaurin polynomial** for f to be

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k$$

Example 1

Find the Maclaurin polynomials p_0, p_1, p_2, p_3 and p_n for e^x.

<u>Solution</u>:

Here $f(x) = e^x$.

To help us find these polynomials, we will construct the following table:

Example 1 (continued)

n =Term #	$f^{(n)}(x)$	$f^{(n)}(0)$	$\frac{f^{(n)}(0)}{n!}$	$\frac{f^{(n)}(x)}{n!}x^n$
0	$f(x) = e^x$	$f(0) = 1$	$\frac{1}{0!} = 1$	1
1	$f'(x) = e^x$	$f'(0) = 1$	$\frac{1}{1!} = 1$	x
2	$f''(x) = e^x$	$f''(0) = 1$	$\frac{1}{2!} = \frac{1}{2}$	$\frac{1}{2}x^2$
3	$f'''(x) = e^x$	$f'''(0) = 1$	$\frac{1}{3!} = \frac{1}{6}$	$\frac{1}{6}x^3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$f^{(n)}(x) = e^x$	$f^{(n)}(0) = 1$	$\frac{1}{n!}$	$\frac{1}{n!}x^n$

Example 1 (continued)

Using our table, we find:

$$p_0(x) = f(0) = 1$$

$$p_1(x) = f(0) + f'(0)x = 1 + x$$

$$p_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 1 + x + \frac{x^2}{2}$$

$$p_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

$$\vdots$$

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!}$$

n-th Taylor Polynomial

(To approximate $f(x)$ by a polynomial on an interval centered at $x = a$.)

If f can be differentiated n times at $x = a$, then we define the ***n*-th Taylor polynomial** for f about $x = a$ to be

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Example 2

Find the Taylor polynomials p_0, p_1, p_2, p_3 for $\sin x$ about $x = \frac{\pi}{3}$.

<u>Solution</u>:

Here $f(x) = \sin x$.

To help us find these polynomials, we will construct the following table:

Example 2 (continued)

$n =$ Term #	$f^{(n)}(x)$	$f^{(n)}\left(\frac{\pi}{3}\right)$	$\frac{f^{(n)}\left(\frac{\pi}{3}\right)}{n!}$	$\frac{f^{(n)}\left(\frac{\pi}{3}\right)}{n!}\left(x-\frac{\pi}{3}\right)^n$
0	$f(x) = \sin x$	$f\left(\frac{\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$	$\frac{(\sqrt{3}/2)}{0!} = \frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$
1	$f'(x) = \cos x$	$f'\left(\frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$	$\frac{(1/2)}{1!} = \frac{1}{2}$	$\frac{1}{2}\left(x-\frac{\pi}{3}\right)$
2	$f''(x) = -\sin x$	$f''\left(\frac{\pi}{3}\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$	$-\frac{(\sqrt{3}/2)}{2!} = -\frac{\sqrt{3}}{4}$	$-\frac{\sqrt{3}}{4}\left(x-\frac{\pi}{3}\right)^2$
3	$f'''(x) = -\cos x$	$f'''\left(\frac{\pi}{3}\right) = -\cos\frac{\pi}{3} = -\frac{1}{2}$	$-\frac{(1/2)}{3!} = -\frac{1}{12}$	$-\frac{1}{12}\left(x-\frac{\pi}{3}\right)^3$

Example 2 (continued)

Using our table, we find:

$$p_0(x) = f\left(\frac{\pi}{3}\right)$$
$$= \frac{\sqrt{3}}{2}$$

$$p_1(x) = f\left(\frac{\pi}{3}\right) + f'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right)$$
$$= \frac{\sqrt{3}}{2} + \frac{1}{2}\left(x - \frac{\pi}{3}\right)$$

Example 2 (continued)

$$p_2(x) = f\left(\frac{\pi}{3}\right) + f'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right) + \frac{f''\left(\frac{\pi}{3}\right)}{2!}\left(x - \frac{\pi}{3}\right)^2$$
$$= \frac{\sqrt{3}}{2} + \frac{1}{2}\left(x - \frac{\pi}{3}\right) - \frac{\sqrt{3}}{4}\left(x - \frac{\pi}{3}\right)^2$$

$$p_3(x) = f\left(\frac{\pi}{3}\right) + f'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right) + \frac{f''\left(\frac{\pi}{3}\right)}{2!}\left(x - \frac{\pi}{3}\right)^2$$
$$+ \frac{f'''\left(\frac{\pi}{3}\right)}{3!}\left(x - \frac{\pi}{3}\right)^3$$
$$= \frac{\sqrt{3}}{2} + \frac{1}{2}\left(x - \frac{\pi}{3}\right) - \frac{\sqrt{3}}{4}\left(x - \frac{\pi}{3}\right)^2 - \frac{1}{12}\left(x - \frac{\pi}{3}\right)^3$$

Maclaurin and Taylor Series

As $n \to \infty$, we hope that the n-th Maclaurin/Taylor polynomials will converge to $f(x)$.

If f has derivatives of all orders at $x = a$, then we define the **Taylor series** for f about $x = a$ to be

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

If $a = 0$, then this is called the **Maclaurin series** for f and is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k$$

Example 3

We saw in Example 1 that the n-th Maclaurin polynomial for e^x is

$$p_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} = \sum_{k=0}^{n} \frac{x^k}{k!}$$

So the Maclaurin series for e^x is

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Example 4

Find the Taylor series for $\frac{1}{x}$ about $x = 1$.

<u>Solution</u>:

Here $f(x) = \frac{1}{x}$.

To help us find this series, we will construct the following table:

Example 4 (continued)

$n =$ Term #	$f^{(n)}(x)$	$f^{(n)}(1)$	$\frac{f^{(n)}(1)}{n!}$	$\frac{f^{(n)}(1)}{n!}(x-1)^n$
0	$\frac{1}{x}$	1	$\frac{1}{0!} = 1$	1
1	$-1 \cdot \frac{1}{x^2}$	-1	$\frac{-1}{1!} = -1$	$-(x-1)$
2	$-1(-2)\frac{1}{x^3}$	2	$\frac{2}{2!} = 1$	$(x-1)^2$
3	$-1(-2)(-3)\frac{1}{x^4}$	-6	$\frac{-6}{3!} = -1$	$-(x-1)^3$
$\vdots$				
n	$-1(-2)(-3)\cdots(-n)\frac{1}{x^{n+1}}$ $= (-1)^n(n!)\frac{1}{x^{n+1}}$	$(-1)^n(n!)$	$\frac{(-1)^n(n!)}{n!}$ $= (-1)^n$	$(-1)^n(x-1)^n$

Example 4 (continued)

Using the table, we see that the Taylor series for $\frac{1}{x}$ about $x = 1$ is

$$\sum_{k=0}^{\infty} (-1)^k (x-1)^k$$

Naming the Maclaurin/Taylor Series

Brook Taylor (British) did not invent Taylor series, and Maclaurin series were not developed by Colin Maclaurin (Scottish).

James Gregory (Scottish) was working with Taylor series when Taylor was only a few years old. Gregory also published the Maclaurin series for many trigonometric functions ten years before Maclaurin was born.

Taylor was not aware of Gregory's work when he published his book *Methodus incrementorum directa et inversa*, which contained what we now call Taylor series. Maclaurin quoted Taylor's work in a calculus book he wrote in 1742. Maclaurin's book popularized series representations of functions, and although Maclaurin never claimed to have discovered them, Taylor series centered at $a = 0$ later became known as Maclaurin series.

History balanced things in the end. Maclaurin, a brilliant mathematician, was the original discoverer of the rule for solving systems of equations that we now call Cramer's rule.