Convergence of Taylor Series

Part 1

n-th Remainder

If we approximate f by its n-th Taylor polynomial about $x = a$, $p_n(x)$,

then the error at x is

$$R_n(x) = f(x) - p_n(x)$$

$R_n(x)$ is called the **$\boldsymbol{n}$-th remainder**.

Taylor's Theorem

Suppose f can be differentiated $n+1$ times at each point in an interval containing the point a.

Then for each x in the interval, there is at least one point c between a and x such that

$$R_n(x) = f(x) - p_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Taylor's Formula With Remainder

$$f(x) = p_n(x) + R_n(x)$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$+ \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Side Note

The n-th remainder formula

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

was not due to Taylor, but rather to Lagrange.

For this reason, it is frequently called **Lagrange's form of the remainder**.

Theorem

The equality

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

holds if and only if

$$\lim_{n\to\infty} R_n(x) = 0.$$

Example 1

Show that the Maclaurin series for e^x converges to e^x for all x.

Solution:

The Maclaurin series for e^x is

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Example 1 (continued)

Taylor's Formula says:

$$f(x) = p_n(x) + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Using $f(x) = e^x$ and $a = 0$ we get,

$$e^x \ = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{e^c}{(n+1)!}x^{n+1}$$

where c is between 0 and x.

Example 1 (continued)

We must show that $\lim_{n\to\infty} R_n(x) = 0$.

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \frac{e^c}{(n+1)!} x^{n+1}.$$

Since $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges for all x, we know

$$\lim_{k\to\infty} \frac{x^k}{k!} = 0.$$

Now we will consider 3 cases:

Example 1 (continued)

Case 1: $x > 0$

$$0 < c < x$$
$$0 < e^c < e^x$$
$$0 < e^c \frac{x^{n+1}}{(n+1)!} < e^x \frac{x^{n+1}}{(n+1)!}$$
$$0 < R_n(x) < e^x \frac{x^{n+1}}{(n+1)!}$$

Since

$$\lim_{n\to\infty} e^x \cdot \frac{x^{n+1}}{(n+1)!} = e^x \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0$$

the Pinching Theorem tells us

$$\lim_{n\to\infty} R_n(x) = 0$$

Example 1 (continued)

Case 2: $x < 0$

$$x < c < 0$$
$$0 < e^c < 1$$
$$0 < e^c \left|\frac{x^{n+1}}{(n+1)!}\right| < \left|\frac{x^{n+1}}{(n+1)!}\right|$$
$$0 < |R_n(x)| < \left|\frac{x^{n+1}}{(n+1)!}\right|$$

Since

$$\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0$$

the Pinching Theorem tells us that $\lim_{n\to\infty} |R_n(x)| = 0$ and therefore $\lim_{n\to\infty} R_n(x) = 0$.

Example 1 (continued)

<u>Case 3: $x = 0$</u>

Convergence is obvious in this case since

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + 0 + 0 + 0 + \cdots$$

The Remainder Estimation Theorem

If there exists a positive constant M such that

$$\left|f^{(n+1)}(t)\right| \leq M, \qquad t \text{ between } x \text{ and } a$$

then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

Example 2

For approximately what values of x can you replace $\sin x$ by $x - {x^3}/_6$ with an error of magnitude no greater than 2 x 10^{-4}?

<u>Solution</u>:

Here $f(x) = \sin x$.

All of the derivative are $\pm \sin x$ or $\pm \cos x$.

Example 2 (continued)

So

$$\left|f^{(n+1)}(t)\right| \leq 1 = M \text{ for all } x$$

By the Remainder Estimation Theorem

$$|R_n(x)| \leq \frac{1}{(n+1)!}|x|^{n+1}$$

Recall that

$$x - {x^3}/{6}$$

Is the 3-rd Maclaurin polynomial of $\sin x$.

So we will choose $n = 3$.

Example 2 (continued)

$$|R_n(x)| \leq \frac{1}{(n+1)!}|x|^{n+1}$$

$$|R_3(x)| \leq \frac{1}{(3+1)!}|x|^{3+1}$$

$$|R_3(x)| \leq \frac{1}{24}|x|^{4}$$

and we want

$$|R_3(x)| \leq 2 \times 10^{-4}$$

Example 2 (continued)

$$|R_3(x)| \leq \frac{1}{24}|x|^4 \leq 2 \times 10^{-4}$$

Solving for x we get:

$$\frac{1}{24}|x|^4 \leq 2 \times 10^{-4}$$

$$|x|^4 \leq 48 \times 10^{-4}$$

$$|x| \leq \sqrt[4]{48} \times 10^{-1} \approx 0.26321$$

Example 2 (continued)

So the answer to:

For approximately what values of x can you replace $\sin x$ by $x - {x^3}/{6}$ with an error of magnitude no greater than 2 x 10^{-4}?

is:

$$|x| \leq \sqrt[4]{48} \times 10^{-1} \approx 0.26321$$

Frequently used Maclaurin Series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \qquad |x| < \infty$$

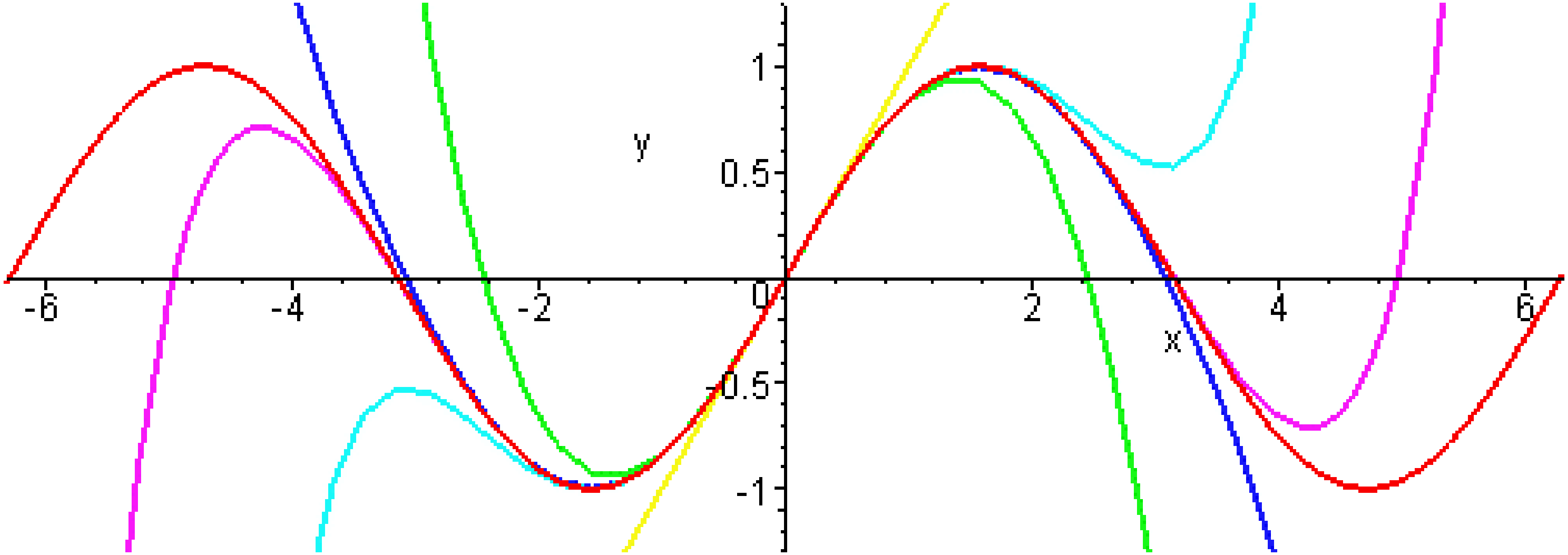
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \qquad -1 < x \leq 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \qquad |x| \leq 1$$

Q: Why do truncated Maclaurin Series fit the original function so well?

A: Because they are "Taylor" made.

http://web2.uwindsor.ca/math/hlynka/mathfun.html



http://www.peterstone.name/Maplepgs/taylor.html