COUNTING THINGS

ORDERED AND NOT ORDERED, WITH AND WITHOUT REPLACEMENT

Combinatorics is "the science of counting". Combinatorialists have come up with many handy formulas to count many handy things. Your textbook discusses some of these formulas; this handout expands on that discussion.

$n!$: This is the number of ways to rearrange n items **ordered without replacement**. To read it out loud, you say "n factorial". We use the shorthand $n!$ for $n \times (n-1) \times (n-2) \times \ldots \times 1$, or in other words, to multiply all the numbers from n on down.

A common setting where you would use this type of formula is in rearrangements. For example, ways a deck of cards could be shuffled (which card is on top, second from the top, third from the top...), or students could arrive to the classroom (who is earliest, who is next earliest...). All options are available for the first choice, but once the first has been chosen it can't be used for the second, the first and second can't be third, etc.

One way to see why this formula makes sense is to think about writing down which item you select each time; there are thus n blanks on your list, and each blank can be filled with any of n options for the first, but then the choices decrease. For example, when you are keeping track of shuffles of a deck of 52 cards, you have 52 choices for the top card on the pile. Once that card is decided, there are only 51 cards left that could possibly be the second one on the pile. Continuing this process gives us $52 \times 51 \times 50 \times \ldots \times 1$ options overall.

n^k: This is the number of ways to choose k items out of n **ordered with replacement**.

A common setting where you would use this type of formula is in repeated tries of the same thing, where the number of options doesn't change. For example, rolling dice over and over, or flipping a coin over and over, or making up nonsense words one letter at a time. The die can come up any one of 6 numbers each time, the coin can come up heads or tails each time, the next letter in the word can be any one of the 26 each time.

One way to see why this formula makes sense is because of the **Rule Of Products**. There are n options for each item, and there are k items, so $n \times n \times n \times \ldots \times n = n^k$ is the correct number.

$(n)_k$**:** This is the number of ways to choose k items out of n **ordered without replacement**. To read it out loud, you say "n falling factorial k". We use the shorthand $(n)_k$ for $n \times (n-1) \times (n-2) \ldots \times (n-k+1)$, or in other words, multiply the k numbers from n on down. Another way to write it is $n!/(n-k)!$.

A common setting where you would use this type of formula is in repeated tries of the same thing, where the number of options changes as you go. For example, dealing out a few cards one at a time, or picking first, second, and third place winners. All options are available for the first choice, but once the first has been chosen it can't be used for the second, the first and second can't be third, etc.

One way to see why this formula makes sense is to think about writing down which item you select each time; there are thus r blanks on your list, and each blank can be filled with any of n options for the first, but then the choices decrease. For example, when you are selecting first, second, and third winners out of players $\{a, b, c, d, e\}$, you have 5 choices for the first place winner. Once that person is picked (pretend that it is a), there are only four people left who could be the second place winner (in our pretend example, they are b, c, d, e). Continuing this process gives us $5 \times 4 \times 3$ options overall.

$\binom{n}{k}$**:** This is the number of ways to choose k items out of n **unordered without replacement**. To read it out loud, you say "n choose k". We use the shorthand $\binom{n}{k}$ for $(n)_k/k!$. Another way to write it is $n!/(n-k)!k!$.

A common setting where you would use this type of formula is if you are picking a group of things from a list of single options, but you don't care what was picked first. For example, a hand of cards in bridge, or a study group of classmates. You can't have two ace of hearts in your hand, and you can't have two clones in your study group, so it's "without replacement". You don't care which card was dealt first, and you don't care who arrived to the study room first, so it's "unordered".

One way to see why this formula works is to think about it in terms of $(n)_k$. You start by picking k options without replacement. But that number is way too big! For example, that counts the study group {Angela, Brooke, Conor} as different from {Brooke, Angela, Conor}, even though we know they are really the same people. In fact, the ABC study group would be counted six times, one time for each ordering ABC, BAC, ACB, CBA, BCA, CAB. So you need to divide by the number of orderings, which is $k!$.