Homework 1 Solution

1. (a) If $U_1 = 2Y - 1$, then

$$F_{U_1}(u_1) = P(U_1 \le u_1) = P(2Y-1 \le u_1) = P\left(Y \le \frac{u_1+1}{2}\right) = \int_{\{Y \le (u_1+1)/2\}} f_Y(y)\,dy$$
$$= \int_0^{(u_1+1)/2} 2(1-y)\,dy = 1 - \left(1 - \frac{u_1+1}{2}\right)^2 = 1 - \left(\frac{1-u_1}{2}\right)^2$$

so that

$$f_{U_1}(u_1) = \frac{d}{du_1}F_{U_1}(u_1) = \frac{d}{du_1}\left[1 - \left(\frac{1-u_1}{2}\right)^2\right] = \frac{1-u_1}{2}, \quad -1 < u_1 < 1$$

(b) If $U_2 = 1 - 2Y$, then

$$F_{U_2}(u_2) = P(U_2 \le u_2) = P(1-2Y \le u_2) = P\left(Y \ge \frac{1-u_2}{2}\right) = \int_{\{Y \ge (1-u_2)/2\}} f_Y(y)\,dy$$
$$= \int_{(1-u_2)/2}^{1} 2(1-y)\,dy = \left(1 - \frac{(1-u_2)}{2}\right)^2 = \left(\frac{1+u_2}{2}\right)^2$$

so that

$$f_{U_2}(u_2) = \frac{d}{du_2}F_{U_2}(u_2) = \frac{d}{du_2}\left(\frac{1+u_2}{2}\right)^2 = \frac{1+u_2}{2}, \quad -1 < u_2 < 1$$

(c) If $U_3 = Y^2$ with $0 < Y < 1$, then $P(U_3 \le u_3) = P(Y^2 \le u_3) = P(Y \le \sqrt{u_3})$, and hence

$$F_{U_3}(u_3) = P(U_3 \le u_3) = P(Y^2 \le u_3) = P(Y \le \sqrt{u_3}) = \int_0^{\sqrt{u_3}} 2(1-y)\,dy = 1 - (1-\sqrt{u_3})^2$$

so that

$$f_{U_3}(u_3) = \frac{d}{du_3}F_{U_3}(u_3) = \frac{d}{du_3}[1-(1-\sqrt{u_3})^2] = 2(1-\sqrt{u_3})\frac{1}{2\sqrt{u_3}} = \frac{1}{\sqrt{u_3}} - 1, \quad 0 < u_3 < 1$$

2. (a) If $U_1 = 2Y - 1 = h(Y)$, then $Y = (U_1+1)/2 = h^{-1}(U_1)$ and $\frac{dy}{du_1} = 1/2$. Hence,

$$f_{U_1}(u) = f_Y(y)\left|\frac{dy}{du_1}\right| = 2(1-y)\left|\frac{1}{2}\right| = 2\left(1 - \frac{u_1+1}{2}\right)\frac{1}{2} = \frac{1-u_1}{2}, \quad -1 < u_1 < 1$$

(b) If $U_2 = 1 - 2Y = h(Y)$, then $Y = (1-U_2)/2 = h^{-1}(U_2)$ and $\frac{dy}{du_2} = -1/2$. Hence,

$$f_{U_2}(u) = f_Y(y)\left|\frac{dy}{du_2}\right| = 2(1-y)\left|-\frac{1}{2}\right| = 2\left(1 - \frac{1-u_2}{2}\right)\frac{1}{2} = \frac{1+u_2}{2}, \quad -1 < u_2 < 1$$

(c) If $U_3 = Y^2 = h(Y)$, note that the pdf method is applicable since $Y \geq 0$. Then we get $Y = \sqrt{U_3} = h^{-1}(U_3)$ and $\frac{dy}{du_3} = 1/(2\sqrt{u_3})$. Hence,

$$f_{U_3}(u) = f_Y(y)\left|\frac{dy}{du_3}\right| = 2(1-y)\left|\frac{1}{2\sqrt{u_3}}\right| = 2(1-\sqrt{u_3})\frac{1}{2\sqrt{u_3}} = \frac{1}{\sqrt{u_3}} - 1, \quad 0 < u_3 < 1$$

3. If Y is uniformly distributed over (-1, 1), then $f_Y(y) = 1/2$, $-1 < y < 1$.

(a) Since $P(|Y| > 1/2) = P(Y < -1/2) + P(Y > 1/2)$ and $f_Y(y) = 1/2$, it follows easily that

$$P(|Y| > 1/2) = \frac{1}{2}$$

(b) We cannot use the pdf method here (why?), so that by setting $U = |Y|$, we first get

$$\begin{aligned} F_U(u) &= P(U \leq u) = P(|Y| \leq u) = P(-u \leq Y \leq u) \\ &= P(Y \leq u) - P(Y < -u) = \int_{-1}^{u} \frac{1}{2}\,dy - \int_{-1}^{-u} \frac{1}{2}\,dy = \frac{u}{2} - \frac{1}{2} + \frac{u}{2} + \frac{1}{2} = u \end{aligned}$$

so that

$$f_U(u) = \frac{d}{du}F_U(u) = \frac{d}{du}u = 1, \quad 0 < u < 1$$

4. If Y is an exponential random variable with parameter $\beta = 1$, then $f_Y(y) = e^{-y}$, $y > 0$. Let $U = \ln Y$. By cdf method, we have that

$$F_U(u) = P(U \leq u) = P(\ln Y \leq u) = P(Y \leq e^u) = F_Y(e^u)$$

Now, recall that the cdf of $Y \sim$ Exponential(1) is $F_Y(y) = 1 - e^{-y}$, and so

$$F_U(u) = F_Y(e^u) = 1 - e^{-e^u}$$

and thus

$$f_U(u) = \frac{d}{du}F_U(u) = \frac{d}{du}(1 - e^{-e^u}) = e^{-e^u}e^u, \quad -\infty < u < \infty$$

Since $U = \ln Y$ is monotone (as a function of Y), we may use the pdf method, with $U = h(Y) = \ln Y$ so that $Y = h^{-1}(U) = e^U$ and $\frac{dy}{du} = e^u$, and hence

$$f_U(u) = f_Y(y)\left|\frac{dy}{du}\right| = e^{-e^u}|e^u| = e^{-e^u}e^u, \quad -\infty < u < \infty$$

5. If $c > 0$ is a constant, then $U = cY = h(Y)$ is an increasing function of Y, and we may use the pdf method, with $Y = U/c = h^{-1}(U)$ and $\frac{dy}{du} = 1/c$. Hence,

$$\begin{aligned} f_U(u) = f_Y(y)\left|\frac{dy}{du}\right| &= \frac{1}{\Gamma(\alpha)\beta^\alpha}y^{\alpha-1}e^{-y/\beta}\left|\frac{1}{c}\right| = \frac{1}{\Gamma(\alpha)\beta^\alpha}\left(\frac{u}{c}\right)^{\alpha-1}e^{-(u/c)/\beta}\frac{1}{c} \\ &= \frac{1}{\Gamma(\alpha)(c\beta)^\alpha}u^{\alpha-1}e^{-u/(c\beta)}, \quad u > 0 \end{aligned}$$

so that $U \sim \text{Gamma}(\alpha, c\beta)$.

6. Recall that $f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$, $-\infty < y < \infty$, and since the pdf method cannot be used here (why?), by using the cdf method, we get (see pages 247-248)

$$\begin{aligned} f_U(u) &= \frac{1}{2\sqrt{u}}[f_Y(\sqrt{u}) + f_Y(-\sqrt{u})] = \frac{1}{2\sqrt{u}}\left[\frac{1}{\sqrt{2\pi}}e^{-u/2} + \frac{1}{\sqrt{2\pi}}e^{-u/2}\right] \\ &= \frac{1}{\sqrt{2\pi u}}e^{-u/2} = \frac{1}{2^{1/2}\Gamma(1/2)u^{1/2}}e^{-u/2} = \frac{1}{\Gamma(1/2)2^{1/2}}u^{1/2-1}e^{-u/2}, \quad u > 0 \end{aligned}$$

so that $U \sim \text{Gamma}(1/2, 2) = \chi^2(1)$.

7. Since $U = e^{-Y}$, it follows that $Y = -\ln U$ and $\frac{dy}{du} = -1/u$. So,

$$f_U(u) = f_Y(y)\left|\frac{dy}{du}\right| = \frac{e^{-y}}{(1+e^{-y})^2}\left|-\frac{1}{u}\right| = \frac{u}{(1+u)^2}\frac{1}{u} = \frac{1}{(1+u)^2}$$

where $0 < u < \infty$.

8. If $U = 1 - Y$, then $Y = 1 - U$ and $\frac{dy}{du} = -1$. Hence,

$$\begin{aligned} f_U(u) = f_Y(y)\left|\frac{dy}{du}\right| &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}y^{\alpha-1}(1-y)^{\beta-1}|-1| = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}(1-u)^{\alpha-1}(1-(1-u))^{\beta-1} \\ &= \frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)\Gamma(\alpha)}u^{\beta-1}(1-u)^{\alpha-1}. \quad 0 < u < 1 \end{aligned}$$

so that $U \sim \text{Beta}(\beta, \alpha)$, and that

$$E(U) = \frac{\beta}{\beta+\alpha} \quad \text{and} \quad \text{Var}(U) = \frac{\beta\alpha}{(\beta+\alpha)^2(\beta+\alpha+1)} = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \text{Var}(Y)$$

(Alternatively, we see that $E(U) = E(1-Y) = 1 - E(Y) = 1 - \alpha/(\alpha+\beta) = \beta/(\beta+\alpha)$ and that $\text{Var}(U) = \text{Var}(1-Y) = \text{Var}(Y)$).