Homework 2 Solution

1. The cdf of $U = Y_1 - Y_2$ is given by

$$\begin{aligned} F_U(u) = P(U \le u) = P(Y_1 - Y_2 \le u) &= \iint_{\{Y_1 - Y_2 \le u\}} f_{Y_1,Y_2}(y_1, y_2)\, dy_1 dy_2 \\ &= \int_0^\infty \int_{y_2}^{u+y_2} e^{-y_1}\, dy_1 dy_2 \quad \text{(draw a picture of region)} \\ &= 1 - e^{-u} \end{aligned}$$

where $0 < y_2 < y_1 < \infty \implies 0 < y_1 - y_2 < \infty \implies 0 < u < \infty$. Then,

$$f_U(u) = \frac{d}{du} F_U(u) = \frac{d}{du}(1 - e^{-u}) = e^{-u}, \quad u > 0$$

Here, we recognize that this is the pdf of Exponential(1), and hence $U \sim$ Exponential(1). (Note also that $F_U(u) = 1 - e^{-u}$, $u > 0$ is the cdf of Exponential(1), as well).

2. By independence of Y_1 and Y_2, we have that

$$f_{Y_1,Y_2}(y_1, y_2) = f_{Y_1}(y_1) f_{Y_2}(y_2) = 6y_1(1 - y_1) \cdot 3y_2^2 = 18y_1(1 - y_1)y_2^2, \quad 0 < y_1 < 1,\ 0 < y_2 < 1$$

To obtain the cdf of $U = Y_1Y_2$, we must consider the region $Y_1Y_2 \le u$, $0 < Y_1 < 1$, $0 < Y_2 < 1$. Since the resulting region is a unit square minus the $Y_1Y_2 = u$ curve at the top right corner, it is easier to consider the complement (why?), i.e.,

$$\begin{aligned} F_U(u) = P(U \le u) = P(Y_1Y_2 \le u) = 1 - P(Y_1Y_2 > u) &= 1 - \iint_{\{Y_1Y_2 > u\}} f_{Y_1,Y_2}(y_1, y_2)\, dy_1 dy_2 \\ = 1 - \int_u^1 \int_{u/y_2}^1 18y_1(1 - y_1)y_2^2\, dy_1 dy_2 &= 9u^2 - 8u^3 + 6u^3 \ln u, \quad 0 < u < 1 \end{aligned}$$

Thus, the pdf of U is

$$\begin{aligned} f_U(u) = \frac{d}{du} F_U(u) &= \frac{d}{du}(9u^2 - 8u^3 + 6u^3 \ln u) = 18u - 24u^2 + 18u^2 \ln u + 6u^3 \frac{1}{u} \\ &= 18u - 18u^2 + 18u^2 \ln u = 18u(1 - u - u \ln u), \quad 0 < u < 1 \end{aligned}$$

3. Define $Y_i = \{\text{lifetime of component } i\}$, $i = 1, 2$, where Y_1 and Y_2 are independent. Then $Y_1 \sim$ Exponential(1) and $Y_2 \sim$ Exponential(1), so that $f_{Y_1}(y_1) = e^{-y_1}$ and $f_{Y_2}(y_2) = e^{-y_2}$, with $y_1 > 0$ and $y_2 > 0$, and thus

$$f_{Y_1,Y_2}(y_1, y_2) = f_{Y_1}(y_1) f_{Y_2}(y_2) = e^{-y_1} e^{-y_2} = e^{-(y_1+y_2)}, \quad y_1 > 0,\ y_2 > 0.$$

Now,

$$U = \{\text{average lifetime of the two components}\} = \frac{Y_1 + Y_2}{2}$$

so that using the cdf method

$$\begin{aligned}
F_U(u) &= P(U \le u) = P\left(\frac{Y_1+Y_2}{2} \le u\right) = P(Y_1+Y_2 \le 2u) \\
&= \iint_{Y_1+Y_2 \le 2u} f_{Y_1,Y_2}(y_1,y_2)\, dy_1 dy_2 = \int_0^{2u} \int_0^{2u-y_2} e^{-(y_1+y_2)}\, dy_1 dy_2 \\
&= 1 - e^{-2u} - 2ue^{-2u}
\end{aligned}$$

where $u > 0$, and hence the pdf is

$$f_U(u) = \frac{d}{du} F_U(u) = \frac{d}{du}(1 - e^{-2u} - 2ue^{-2u}) = 4ue^{-2u}, \quad u > 0$$

4. Let $U_1 = (Y_1+Y_2)/2$, and take $U_2 = Y_2$. Then $Y_2 = U_2$ and $Y_1 = 2U_1 - U_2$, so that

$$J = \begin{vmatrix} \frac{\partial y_1}{\partial u_1} & \frac{\partial y_1}{\partial u_2} \\ \frac{\partial y_2}{\partial u_1} & \frac{\partial y_2}{\partial u_2} \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} = 2$$

Then

$$f_{U_1,U_2}(u_1,u_2) = f_{Y_1,Y_2}(y_1,y_2)|J| = e^{-(y_1+y_2)}|2| = 2e^{-2u_1}$$

where $0 < y_1 = 2u_1 - u_2 < \infty$ and $0 < y_2 = u_2 < \infty \implies 0 < u_2 < 2u_1 < \infty$. Thus,

$$f_{U_1}(u_1) = \int f_{U_1,U_2}(u_1,u_2)\, du_2 = \int_0^{2u_1} 2e^{-2u_1}\, du_2 = (2u_1)2e^{-2u_1} = 4u_1e^{-2u_1}$$

for $0 < u_1 < \infty$, matching the pdf of Problem 3.

5. Since $U_1 = Y_1Y_2$ and $U_2 = Y_1/Y_2$, we have that

$$\begin{aligned}
U_1/U_2 = Y_1Y_2/(Y_1/Y_2) = Y_2^2 &\quad\implies\quad Y_2 = \sqrt{U_1/U_2} \\
U_2 = Y_1\sqrt{U_1/U_2} &\quad\implies\quad Y_1 = \sqrt{U_1U_2}
\end{aligned}$$

so that

$$J = \begin{vmatrix} \frac{\partial y_1}{\partial u_1} & \frac{\partial y_1}{\partial u_2} \\ \frac{\partial y_2}{\partial u_1} & \frac{\partial y_2}{\partial u_2} \end{vmatrix} = \begin{vmatrix} \frac{1}{2}\left(\frac{u_2}{u_1}\right)^{1/2} & \frac{1}{2}\left(\frac{u_1}{u_2}\right)^{1/2} \\ \frac{1}{2}\left(\frac{1}{u_1u_2}\right)^{1/2} & -\frac{1}{2}\left(\frac{u_1}{u_2^3}\right)^{1/2} \end{vmatrix} = -\frac{1}{4}\left(\frac{u_2}{u_1}\frac{u_1}{u_2^3}\right)^{1/2} - \frac{1}{4}\left(\frac{u_1}{u_2}\frac{1}{u_1u_2}\right)^{1/2} = -\frac{1}{2}\frac{1}{u_2}$$

Then

$$f_{U_1,U_2}(u_1,u_2) = f_{Y_1,Y_2}(y_1,y_2)|J| = \frac{1}{y_1^2y_2^2}\left|-\frac{1}{2}\frac{1}{u_2}\right| = \frac{1}{(u_1u_2)(u_1/u_2)}\frac{1}{2u_2} = \frac{1}{2u_1^2u_2}$$

Now

$$\begin{aligned}
1 \le Y_1 = U_1U_2 &\quad\implies\quad \frac{1}{U_1} \le U_2 \;\text{ or }\; \frac{1}{U_2} \le U_1 \\
1 \le Y_2 = \frac{U_1}{U_2} &\quad\implies\quad U_2 \le U_1
\end{aligned}$$

so that

$$\frac{1}{u_1} \le u_2 \le u_1$$

6. First, $f_{Y_1}(y_1) = 1$, $0 < y_1 < 1$, and $f_{Y_2}(y_2) = 1$, $0 < y_2 < 1$, and since Y_1 and Y_2 are independent, we have that $f_{Y_1,Y_2}(y_1, y_2) = 1$. If $U_1 = Y_1 + Y_2$ and $U_2 = Y_1/Y_2$, then

$$U_1 = Y_1 + Y_2 = U_2Y_2 + Y_2 = (U_2 + 1)Y_2 \implies Y_2 = \frac{U_1}{U_2 + 1}$$
$$U_2 = \frac{Y_1}{Y_2} = \frac{Y_1}{U_1/(U_2 + 1)} \implies Y_1 = \frac{U_1U_2}{U_2 + 1}$$

and hence

$$J = \begin{vmatrix} \frac{\partial y_1}{\partial u_1} & \frac{\partial y_1}{\partial u_2} \\ \frac{\partial y_2}{\partial u_1} & \frac{\partial y_2}{\partial u_2} \end{vmatrix} = \begin{vmatrix} \frac{u_2}{u_2+1} & \frac{u_1}{(u_2+1)^2} \\ \frac{1}{u_2+1} & -\frac{u_1}{(u_2+1)^2} \end{vmatrix} = -\frac{u_1u_2}{(u_2 + 1)^3} - \frac{u_1}{(u_2 + 1)^3} = -\frac{u_1}{(u_2 + 1)^2}$$

Therefore,

$$f_{U_1,U_2}(u_1, u_2) = f_{Y_1,Y_2}(y_1, y_2)|J| = 1 \cdot \left|-\frac{u_1}{(u_2 + 1)^2}\right| = \frac{u_1}{(u_2 + 1)^2}$$

and

$$0 < Y_1 = \frac{U_1U_2}{U_2 + 1} < 1, \quad 0 < Y_2 = \frac{U_1}{U_2 + 1} < 1$$

implies

$$0 < u_1u_2 < u_2 + 1, \quad 0 < u_1 < u_2 + 1$$

7. If Y_1 and Y_2 are independent and identically distributed exponential random variables with $\beta = 1$, then $f_{Y_1}(y_1) = e^{-y_1}$ and $f_{Y_2}(y_2) = e^{-y_2}$, with $y_1 > 0$ and $y_2 > 0$, so that

$$f_{Y_1,Y_2}(y_1, y_2) = e^{-y_1}e^{-y_2}$$

Suppose that $U_1 = Y_1$, $U_2 = Y_1/Y_2$. Then $Y_1 = U_1$ and $Y_2 = U_1/U_2$, so that

$$J = \begin{vmatrix} \frac{\partial y_1}{\partial u_1} & \frac{\partial y_1}{\partial u_2} \\ \frac{\partial y_2}{\partial u_1} & \frac{\partial y_2}{\partial u_2} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1/u_2 & -u_1/u_2^2 \end{vmatrix} = -\frac{u_1}{u_2^2}$$

Then

$$f_{U_1,U_2}(u_1, u_2) = f_{Y_1,Y_2}(y_1, y_2)|J| = e^{-u_1}e^{-u_1/u_2}\left|-\frac{u_1}{u_2^2}\right| = \frac{u_1}{u_2^2}e^{-u_1(u_2+1)/u_2}$$

where $u_1 > 0$ and $u_2 > 0$.

8. If Y_1 and Y_2 are independent exponential random variables, each having parameter $\beta = 1/\lambda$, then

$$f_{Y_1,Y_2}(y_1, y_2) = \lambda e^{-\lambda y_1}\lambda e^{-\lambda y_2} = \lambda^2 e^{-\lambda(y_1+y_2)}, \quad y_1 > 0,\ y_2 > 0$$

Suppose that $U_1 = Y_1 + Y_2$ and $U_2 = e^{Y_1}$. Then $Y_1 = \ln U_2$ and $Y_2 = U_1 - \ln U_2$, and

$$J = \begin{vmatrix} \frac{\partial y_1}{\partial u_1} & \frac{\partial y_1}{\partial u_2} \\ \frac{\partial y_2}{\partial u_1} & \frac{\partial y_2}{\partial u_2} \end{vmatrix} = \begin{vmatrix} 0 & 1/u_2 \\ 1 & -1/u_2 \end{vmatrix} = \frac{1}{u_2}$$

Thus,

$$f_{U_1,U_2}(u_1, u_2) = f_{Y_1,Y_2}(y_1, y_2)|J| = \lambda^2 e^{-\lambda(y_1+y_2)}\left|\frac{1}{u_2}\right| = \frac{\lambda^2}{u_2}e^{-\lambda u_1}$$

with $u_2 > 1$ and $u_1 > \ln u_2 \implies 1 < u_2 < e^{u_1}$.

9. Since $U_1 = Y_1 + Y_2$ and $U_2 = Y_2$, we have that $Y_1 = U_1 - U_2$ and $Y_2 = U_2$, so that

$$J = \begin{vmatrix} \frac{\partial y_1}{\partial u_1} & \frac{\partial y_1}{\partial u_2} \\ \frac{\partial y_2}{\partial u_1} & \frac{\partial y_2}{\partial u_2} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

Then

$$f_{U_1,U_2}(u_1,u_2) = f_{Y_1,Y_2}(y_1,y_2)|J| = f_{Y_1,Y_2}(u_1-u_2,u_2)|1| = f_{Y_1,Y_2}(u_1-u_2,u_2)$$

for $-\infty < u_2 < u_1 < \infty$. To obtain the marginal density of U_1, we take

$$f_{U_1}(u_1) = \int f_{Y_1,Y_2}(u_1-u_2,u_2)\,du_2$$

but since Y_1 and Y_2 are independent, $f_{Y_1,Y_2}(u_1-u_2,u_2) = f_{Y_1}(u_1-u_2)f_{Y_2}(u_2)$, and so

$$f_{U_1}(u_1) = \int f_{Y_1}(u_1-u_2)f_{Y_2}(u_2)\,du_2$$

10. The pdfs of Y_1, Y_2, and Y_3 are given by

$$f_{Y_1}(y_1) = e^{-y_1}, \quad f_{Y_2}(y_2) = e^{-y_2}, \quad f_{Y_3}(y_3) = e^{-y_3} \quad , y_1 > 0,\ y_2 > 0,\ y_3 > 0$$

and by independence, the joint pdf of (Y_1, Y_2, Y_3) is

$$f_{Y_1,Y_2,Y_3}(y_1,y_2,y_3) = f_{Y_1}(y_1)f_{Y_2}(y_2)f_{Y_3}(y_3) = e^{-y_1}e^{-y_2}e^{-y_3} = e^{-(y_1+y_2+y_3)}$$

for $y_1 > 0$, $y_2 > 0$, $y_3 > 0$. Now, since $U = Y_1 + Y_2 + Y_3$,

$$\begin{aligned}
F_U(u) = P(U \le u) = P(Y_1+Y_2+Y_3 \le u) &= \iiint_{Y_1+Y_2+Y_3 \le u} f_{Y_1,Y_2,Y_3}(y_1,y_2,y_3)\,dy_1dy_2dy_3 \\
&= \int_0^u \int_0^{u-y_3} \int_0^{u-y_2-y_3} e^{-(y_1+y_2+y_3)}\,dy_1dy_2dy_3 = \int_0^u \int_0^{u-y_3} (e^{-(y_2+y_3)} - e^{-u})\,dy_2dy_3 \\
&= \int_0^u (e^{-y_3} - e^{-u} - (u-y_3)e^{-u})\,dy_3 = \int_0^u (e^{-y_3} - e^{-u} - ue^{-u} + y_3e^{-u})\,dy_3 \\
&= 1 - e^{-u} - ue^{-u} - u^2e^{-u} + \frac{u^2}{2}e^{-u} = 1 - e^{-u} - ue^{-u} - \frac{u^2}{2}e^{-u}
\end{aligned}$$

for $0 < u < \infty$ (since $u = y_1 + y_2 + y_3 > 0$), so that the pdf of $U = Y_1 + Y_2 + Y_3$ is given by

$$\begin{aligned}
f_U(u) = \frac{d}{du}F_U(u) &= \frac{d}{du}\left(1 - e^{-u} - ue^{-u} - \frac{u^2}{2}e^{-u}\right) \\
&= e^{-u} + ue^{-u} - e^{-u} - ue^{-u} + \frac{u^2}{2}e^{-u} \\
&= \frac{u^2}{2}e^{-u} \\
&= \frac{1}{\Gamma(3)1^3}u^{3-1}e^{-u/1}, \quad u > 0
\end{aligned}$$

which is the pdf of Gamma$(3,1)$, and thus $Y_1 + Y_2 + Y_3 \sim$ Gamma$(3,1)$.

11. Take $U_1 = Y_1+Y_2+Y_3$, and let $U_2 = Y_2+Y_3$ and $U_3 = Y_3$, so that $Y_1 = U_1-U_2$, $Y_2 = U_2-U_3$, $Y_3 = U_3$. The Jacobian is then

$$J = \begin{vmatrix} \frac{\partial y_1}{\partial u_1} & \frac{\partial y_1}{\partial u_2} & \frac{\partial y_1}{\partial u_3} \\ \frac{\partial y_2}{\partial u_1} & \frac{\partial y_2}{\partial u_2} & \frac{\partial y_2}{\partial u_3} \\ \frac{\partial y_3}{\partial u_1} & \frac{\partial y_3}{\partial u_2} & \frac{\partial y_3}{\partial u_3} \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

so that

$$f_{U_1,U_2,U_3}(u_1,u_2,u_3) = f_{Y_1,Y_2,Y_3}(y_1,y_2,y_3)|J| = e^{-(y_1+y_2+y_3)}|1| = e^{-u_1}$$

for $0 < u_3 < u_2 < u_1 < \infty$ (from $u_3 = y_3 < y_2+y_3 = u_2$ and $u_2 < y_2+y_3 < y_1+y_2+y_3 = u_1$). Hence,

$$\begin{aligned} f_{U_1}(u_1) &= \iint f_{U_1,U_2,U_3}(u_1,u_2,u_3)\,du_2du_3 = \int_0^{u_1}\int_{u_3}^{u_1} e^{-u_1}\,du_2du_3 = \int_0^{u_1}\int_0^{u_2} e^{-u_1}\,du_3du_2 \\ &= \int_0^{u_1} u_2e^{-u_1}\,du_2 = \frac{u_1^2}{2}e^{-u_1}, \quad u_1 > 0 \end{aligned}$$

which matches the pdf $f_U(u)$ in Problem 10 (i.e., Gamma$(3,1)$), wherein we conclude that $Y_1+Y_2+Y_3 \sim \text{Gamma}(3,1)$.

12. If $U_1 = Y_1+Y_2$, $U_2 = Y_1+Y_3$, $U_3 = Y_2+Y_3$, then

$$Y_1 = \frac{U_1+U_2-U_3}{2}, \quad Y_2 = \frac{U_1-U_2+U_3}{2}, \quad Y_3 = \frac{-U_1+U_2+U_3}{2}$$

and

$$J = \begin{vmatrix} \frac{\partial y_1}{\partial u_1} & \frac{\partial y_1}{\partial u_2} & \frac{\partial y_1}{\partial u_3} \\ \frac{\partial y_2}{\partial u_1} & \frac{\partial y_2}{\partial u_2} & \frac{\partial y_2}{\partial u_3} \\ \frac{\partial y_3}{\partial u_1} & \frac{\partial y_3}{\partial u_2} & \frac{\partial y_3}{\partial u_3} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

Then

$$f_{U_1,U_2,U_3}(u_1,u_2,u_3) = f_{Y_1,Y_2,Y_3}(y_1,y_2,y_3)|J| = e^{-(y_1+y_2+y_3)}\left|-\frac{1}{2}\right| = \frac{1}{2}e^{-(u_1+u_2+u_3)/2}$$